Understanding Rare but Catastrophic Events via Heavy-tailed Large Deviations

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Northwestern University

International Workshop on Stress Test and Risk Management

May 28, 2019

Joint work with Mihail Bazhba, Jose Blanchet, Bohan Chen, Bert Zwart

Rare Events

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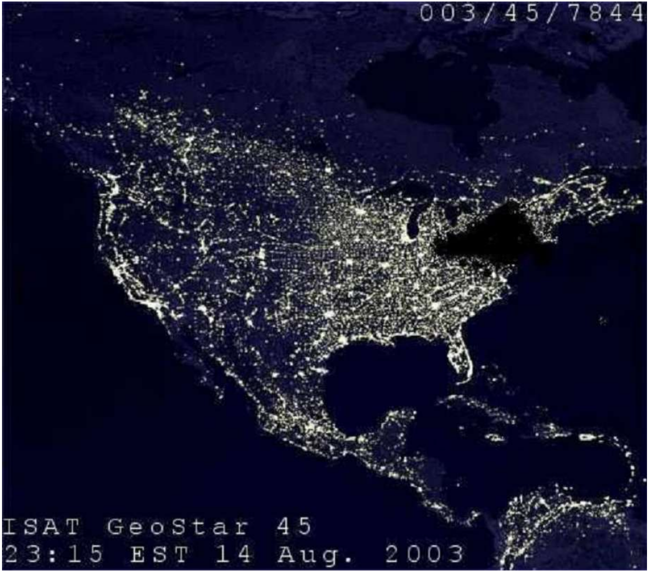

Rare Events

Although rare, rare events matter.

Need for understanding ‘how often?’ & ‘why?’

Rare Events depend on "Tail Behaviors"

Light-Tailed Distributions

- Extreme Values are Very Rare
- Normal, Exponential, etc

Heavy-Tailed Distributions

- Extreme Values are Frequent
- **Power Law**, **Weibull**, etc

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Systemwide rare events

arise because

EVERYTHING goes wrong.

(Conspiracy Principle)

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Systemwide rare events

arise because of

A FEW Catastrophes.

(Catastrophe Principle)

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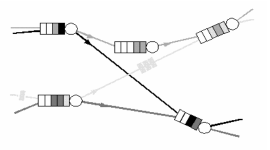
Heavy-tailed rare events are NOT understood well.

But, Heavy Tails are Everywhere:

Computer systems

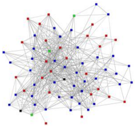


delays, files, ...

Finance

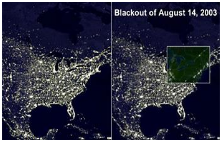
losses

Social networks



popularity, contagion

Energy Systems



blackouts

Goal: Systematic Tools for Heavy-Tailed Systems

In many applications, one can write the rare event of interest as

$$\{\bar{S}_n \in A\}$$

where $\bar{S}_n$ is the whole trajectory of a random walk.

We want to understand $\mathbf{P}(\bar{S}_n \in A)$ and $\mathbf{P}(\bar{S}_n \in \cdot | \bar{S}_n \in A)$

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- $S_k \triangleq X_1 + \cdots + X_k$.

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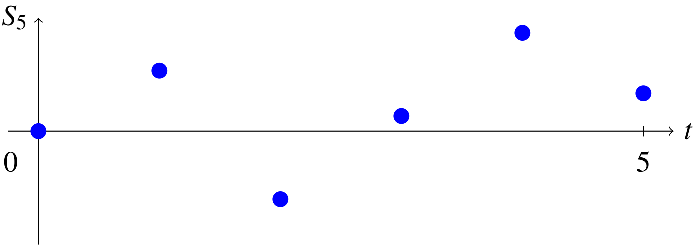

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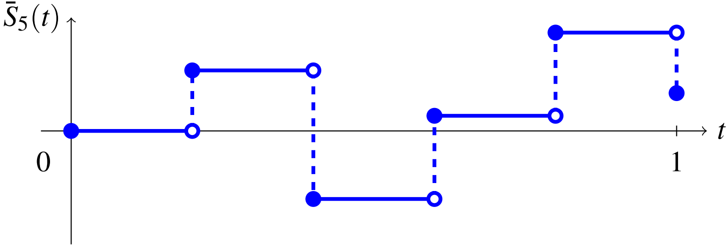

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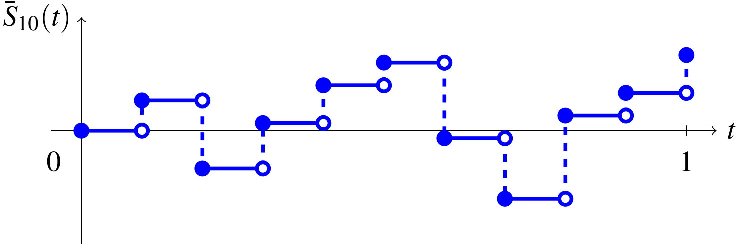

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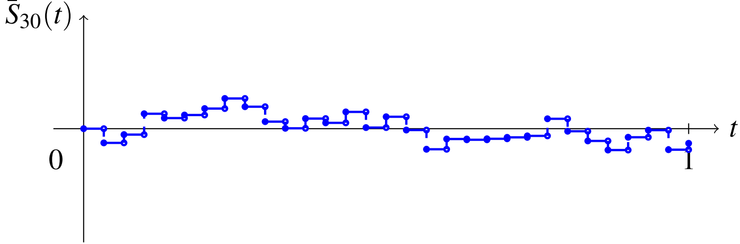

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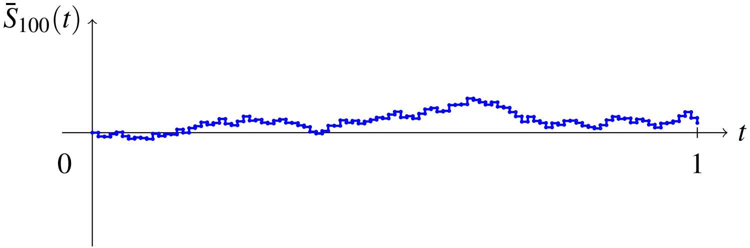

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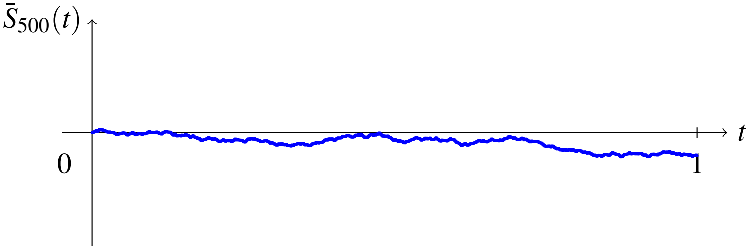

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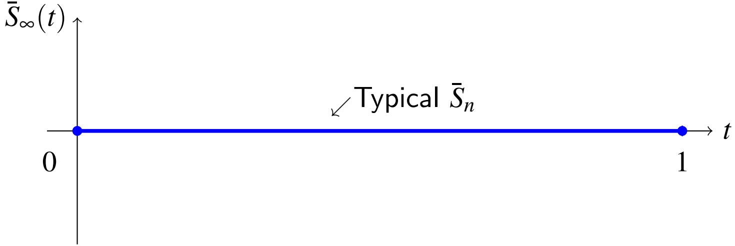

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Outline

Part 1. Large Deviations for Power Law Tails

R., Blanchet, Zwart (2019+)
Annals of Probability

Part 2. Heavy-Tailed Rare Event Simulation

Chen, Blanchet, **R.**, Zwart (2019+)
Mathematics of Operations Research

Part 3. Large Deviations for Weibull Tails

Bazbha, Blanchet, **R.**, Zwart (2017)
Under secound round review at *Annals of Applied Probability*

Part 1. Large Deviations for Power Law Tails

i.e., $\mathbf{P}(X_i \geq x) = x^{-(\alpha+1)},\ \alpha > 0$

What's already known: principle of a single big jump

"In heavy-tailed systems, rare events arise due to one big anomaly."

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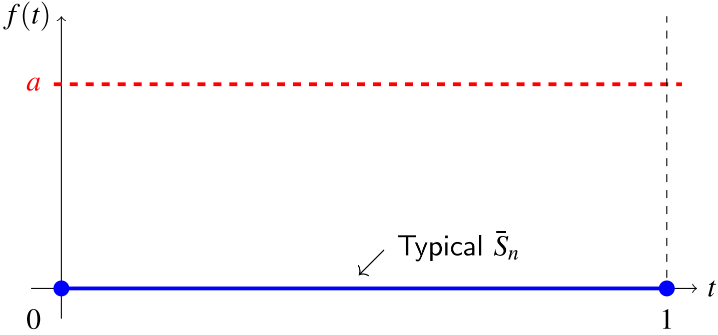

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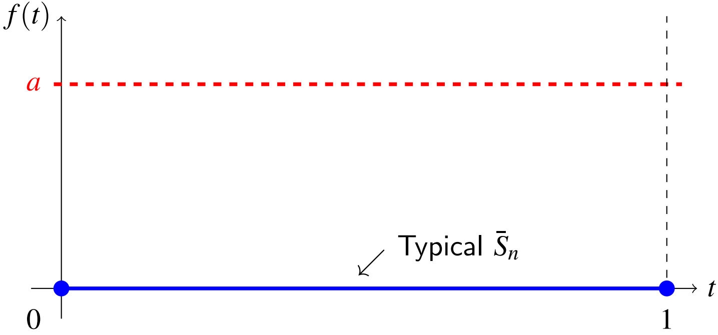

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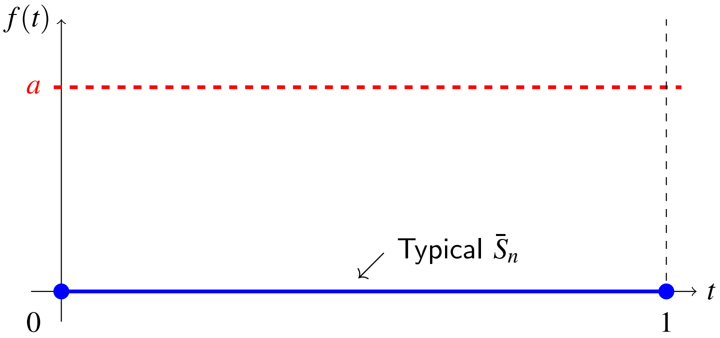

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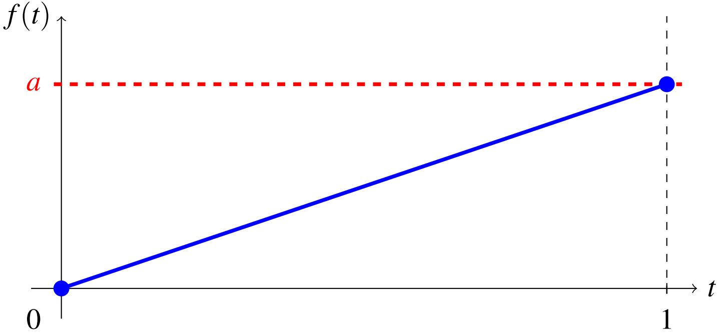

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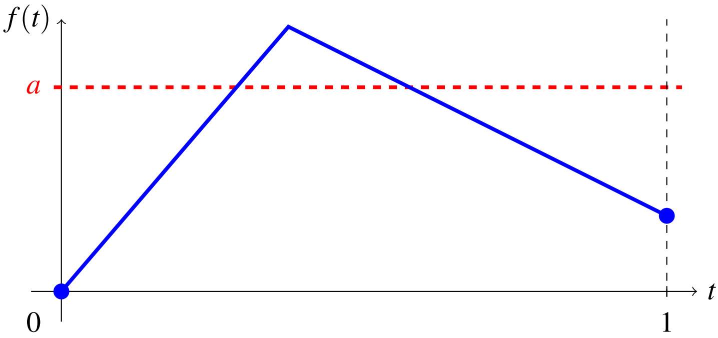

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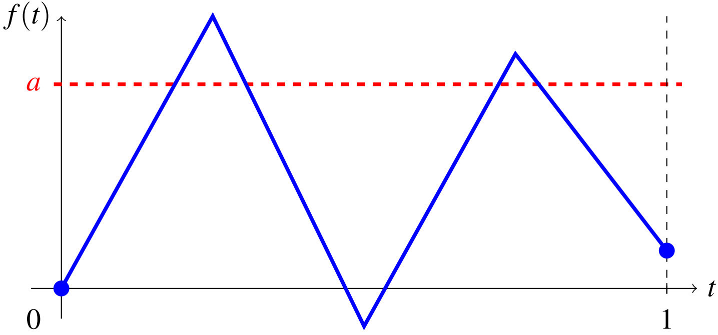

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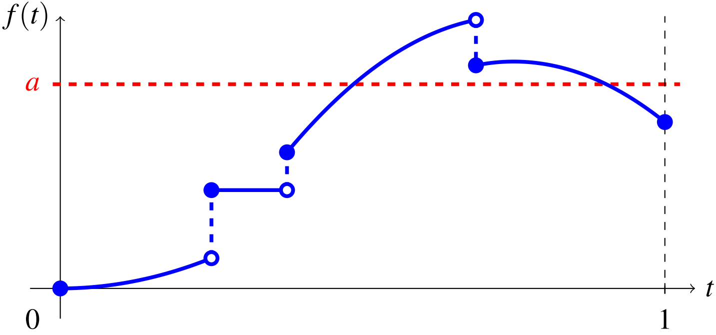

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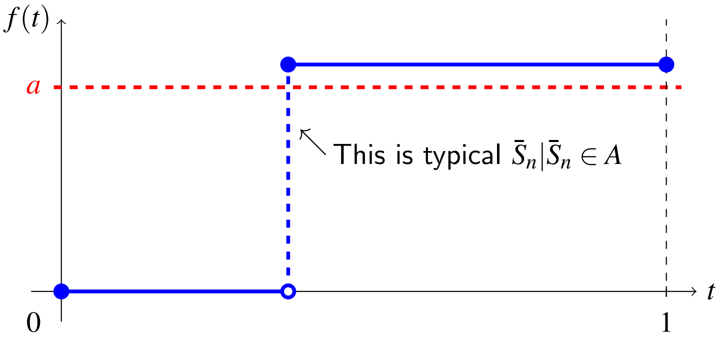

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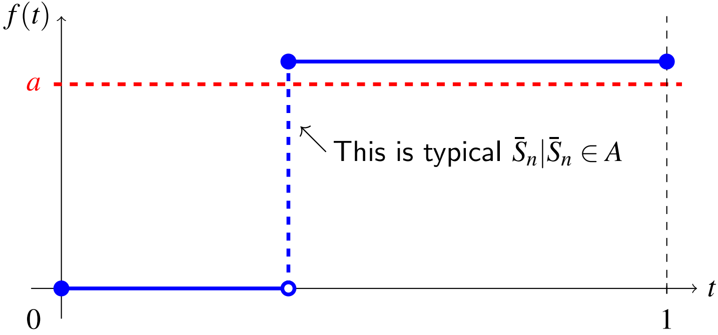

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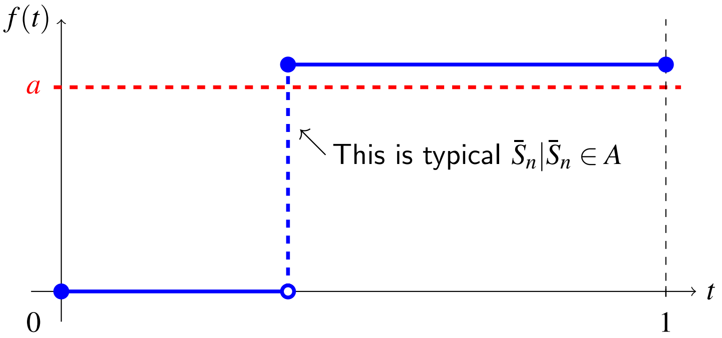

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- $\mathbf{P}(\bar{S}_n \in \cdot | \bar{S}_n \in A) \rightarrow \mathbf{P}(Z \mathbb{1}_{[U \leq t]} \in \cdot)$ for some r.v.-s Z and U

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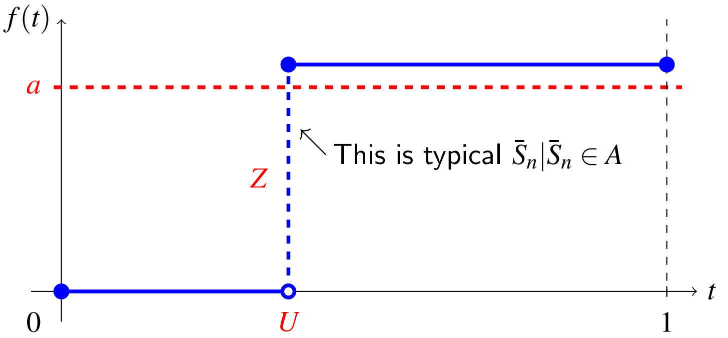

- $A = \{f \in \mathbb{D} : f$ crosses level a on $[0,1]\}$ $\quad$ $\mathbf{P}(X_i \geq x) = x^{-(\alpha+1)}$
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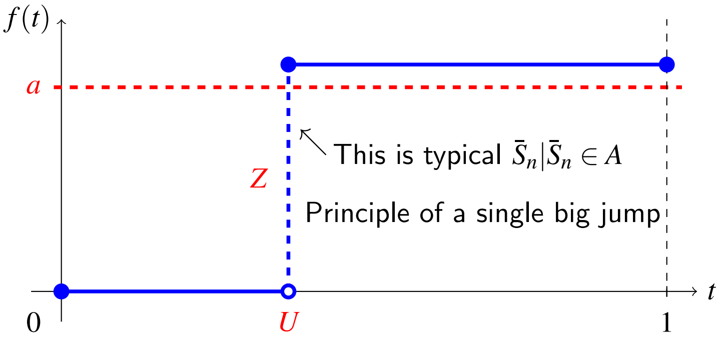

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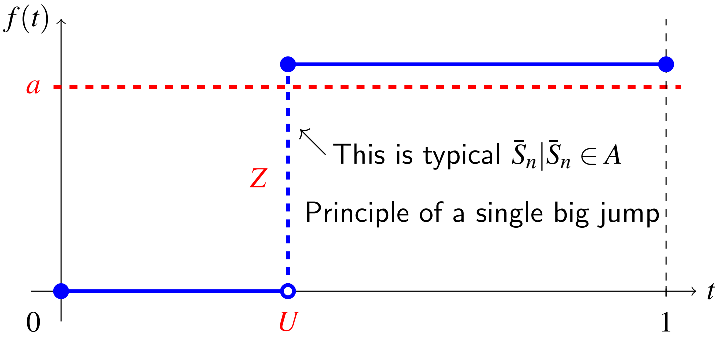

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Hult, Lindskog, Mikosch, Samorodnitsky (2005)

Is everything explained by a single big jump?

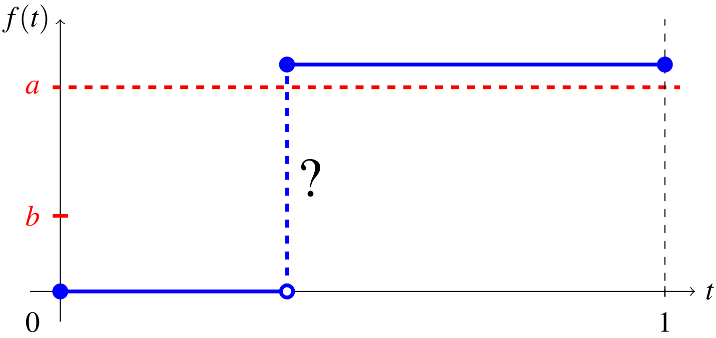
No, by no means, absolutely not:

- Multiple server queues
- Queueing networks
- Re-insured insurance line
- Down-and-in barrier option
- Many more

Is everything explained by a single big jump?

Principle of a single big jump is a very very SPECIAL CASE!

Illustration: What if Large Claims are Reinsured?



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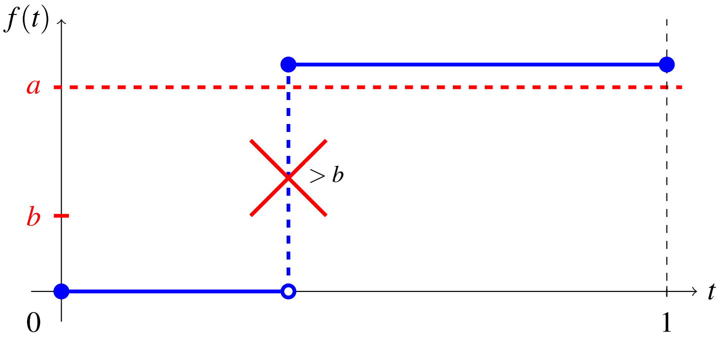

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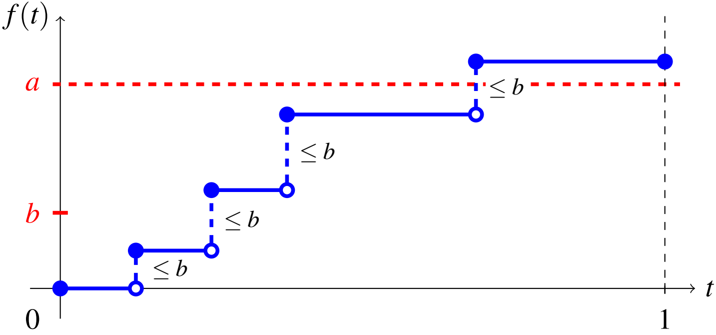

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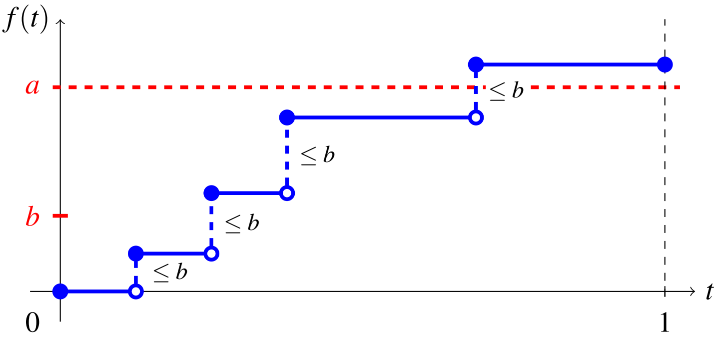

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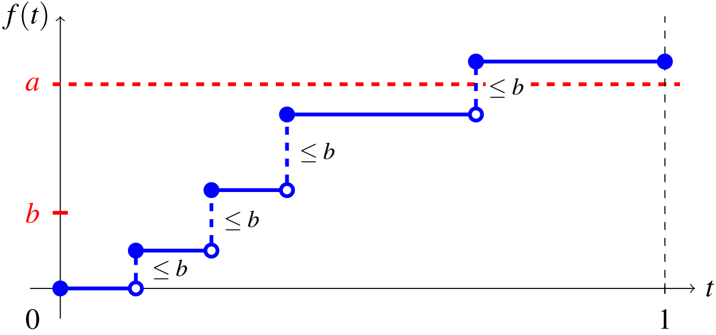

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Exact Asymptotics for Heavy-tailed Random Walks

$\mathbf{P}(X_i \geq x) = x^{-(\alpha+1)}, \quad \alpha > 0$

Theorem (R., Blanchet, Zwart, 2016)

For "general" $A \subseteq \mathbb{D}$

$$C(A^\circ) \leq \liminf_{n\to\infty} \frac{\mathbf{P}(\bar{S}_n \in A)}{n^{-\alpha \mathcal{J}(A)}} \leq \limsup_{n\to\infty} \frac{\mathbf{P}(\bar{S}_n \in A)}{n^{-\alpha \mathcal{J}(A)}} \leq C(A^-).$$

- $\mathcal{J}(A)$: *min #jumps for step functions to be inside A*
- $C(\cdot)$: *a measure*

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LD power index

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Implication: The Principle of Multiple Big Jumps

Under certain regularity conditions on A,

$$\mathcal{L}(\bar{S}_n | \bar{S}_n \in A) \to \mathcal{L}(\bar{S}_{|A})$$

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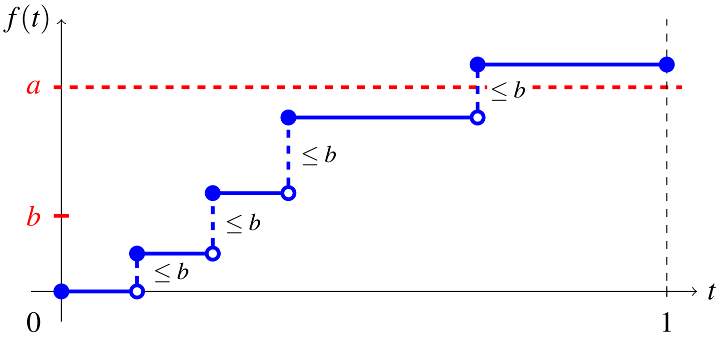
The Principle of $\mathcal{J}(A)$ Big Jumps!

Back to Our Reinsurance Example



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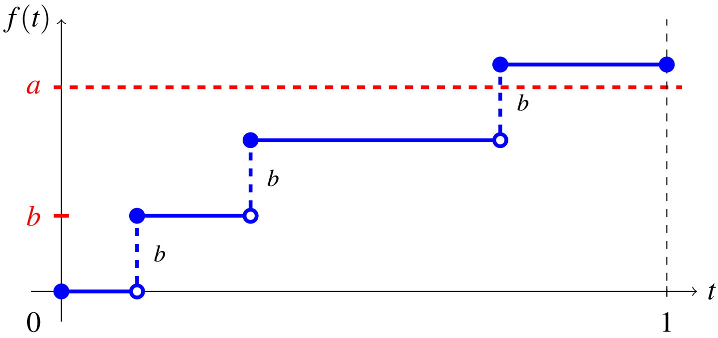

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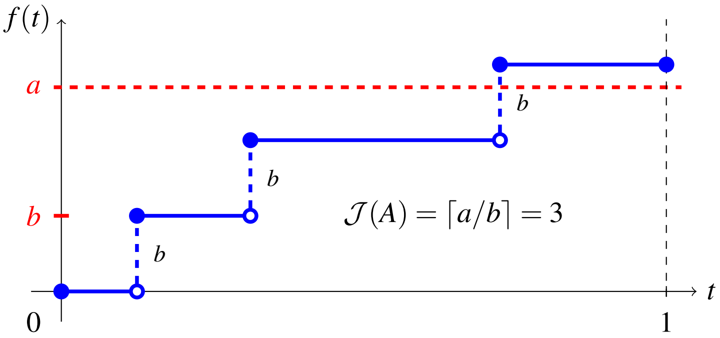

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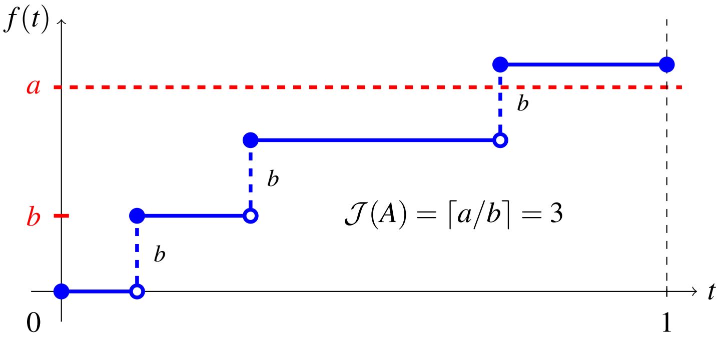

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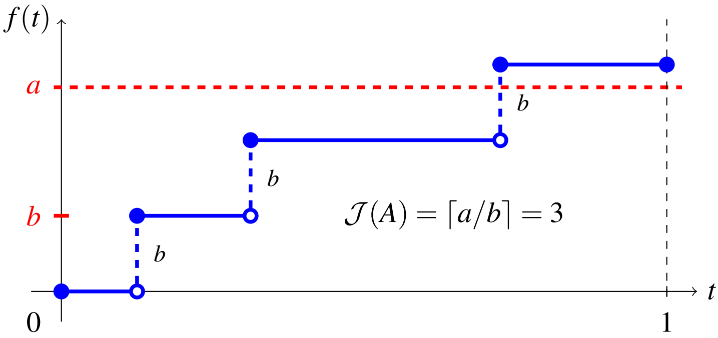

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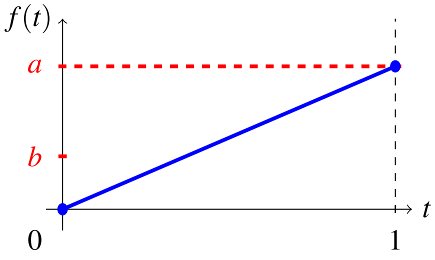
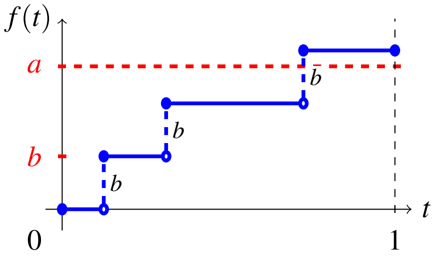


Light-Tailed Claim Size



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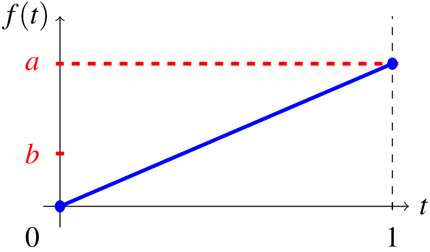

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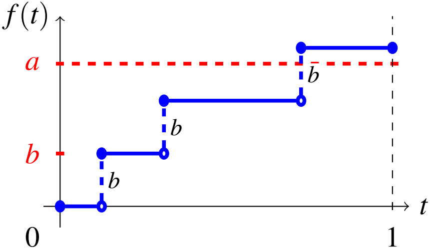

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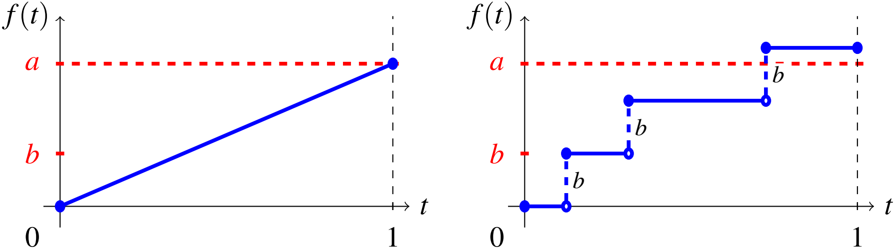

Light-Tailed Claim Size

Heavy-Tailed Claim Size

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Reinsurance helps!

Two-sided Random Walks

$$\mathbf{P}(X_i \geq x) = x^{-(\alpha+1)}, \quad \alpha > 0 \qquad \text{and} \qquad \mathbf{P}(X_i \leq -x) = x^{-(\beta+1)}, \quad \beta > 0.$$

Theorem (R., Blanchet, Zwart, 2016)

For "general" $A \subseteq \mathbb{D}$,

$$\mathbf{P}(\bar{S}_n \in A) \sim n^{-\{\alpha \mathcal{J}(A) + \beta \mathcal{K}(A)\}}.$$

- $\mathcal{J}(A)$: *# of upward jumps*
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of step functions that minimize the cost of staying inside A

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Connection to Impulse Control Problem

The "rate of decay" is determined by a discrete optimization problem:

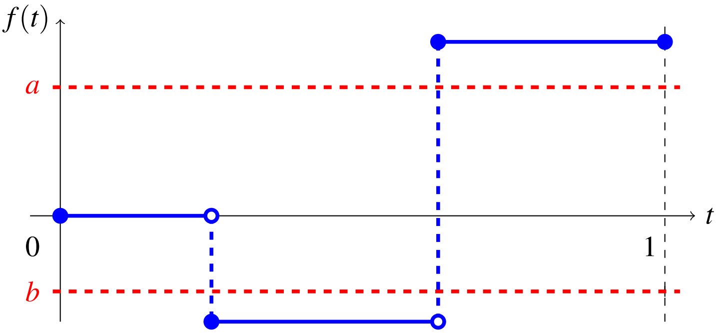
$$
\begin{aligned}
\alpha \mathcal{J}(A) + \beta \mathcal{K}(A) \quad = \quad & \min_{j,k} \; \alpha j + \beta k \\
& \text{subject to } (j,k) \in \mathbb{Z}_+^2 \\
& \qquad \mathbb{D}_{j,k} \cap A \neq \emptyset
\end{aligned}
$$

where

$$\mathbb{D}_{j,k} = \{\text{step functions w/ } j \text{ upward jumps and } k \text{ downward jumps}\}$$

Different from variational calculus that arises in light-tailed cases!

Example: Barrier Option



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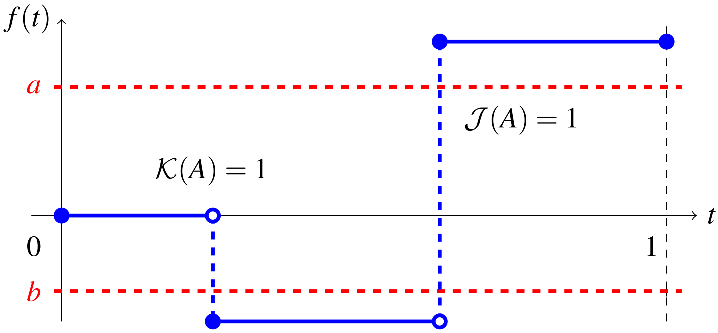

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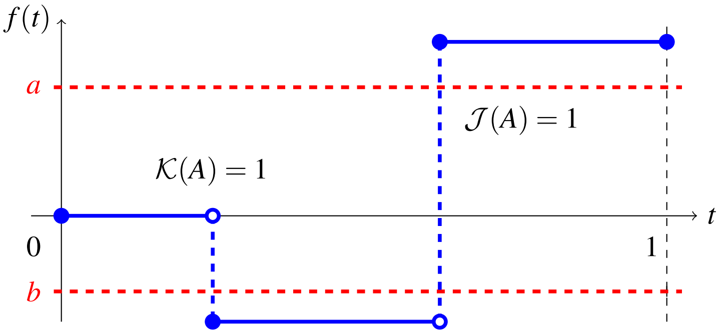

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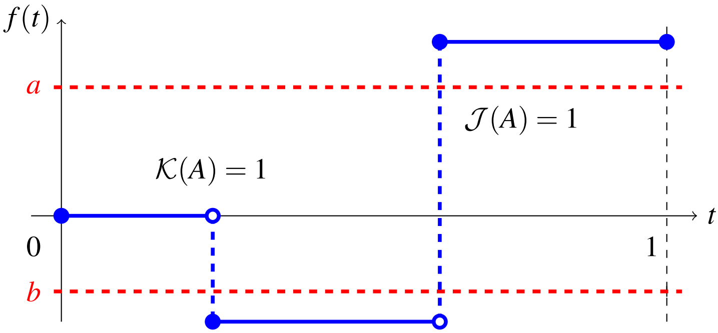

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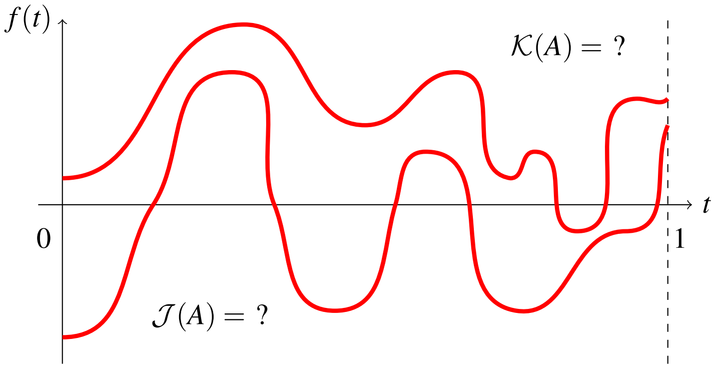
Example: Sausage



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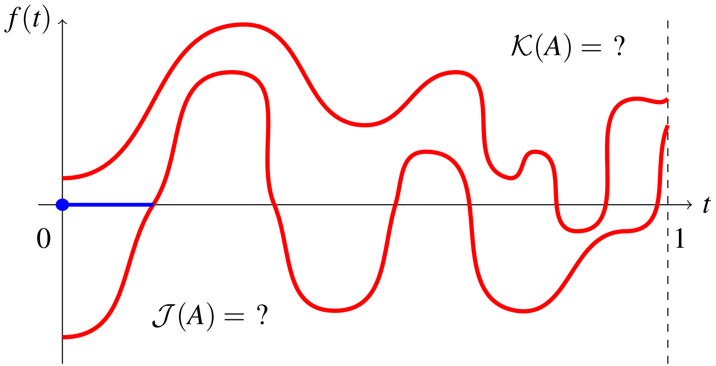

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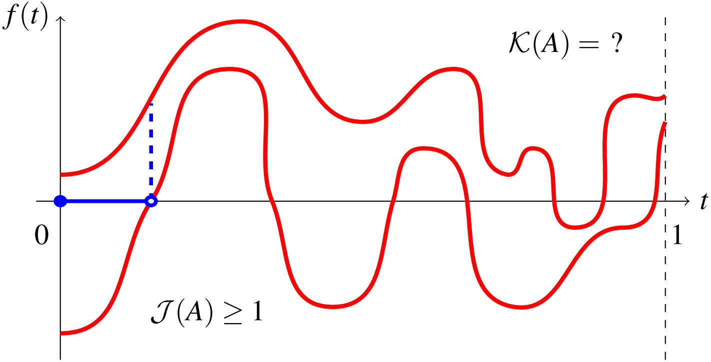

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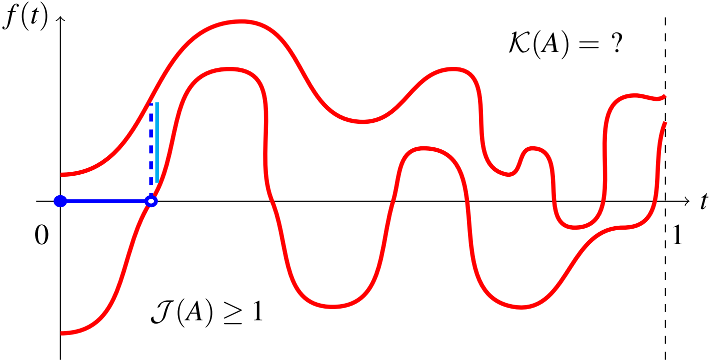

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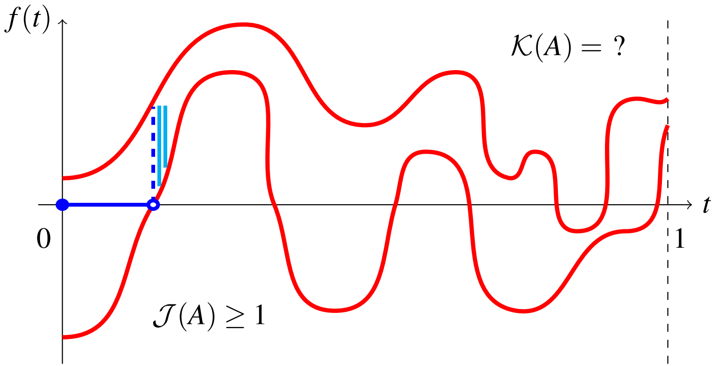

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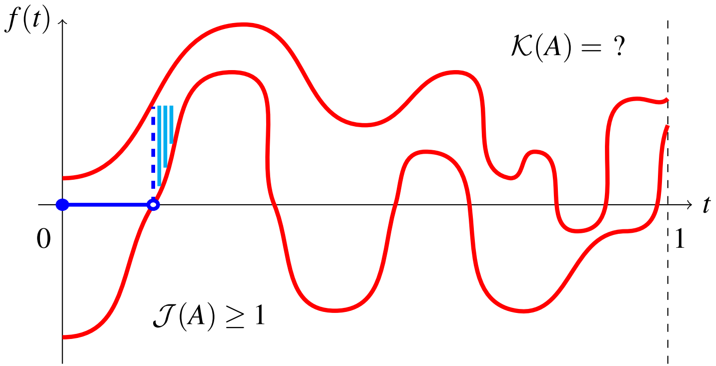

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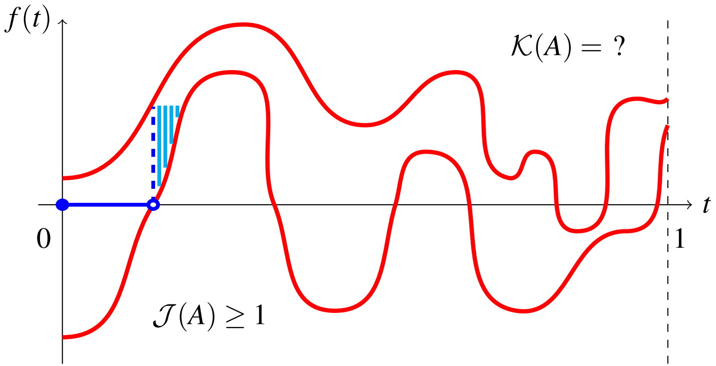

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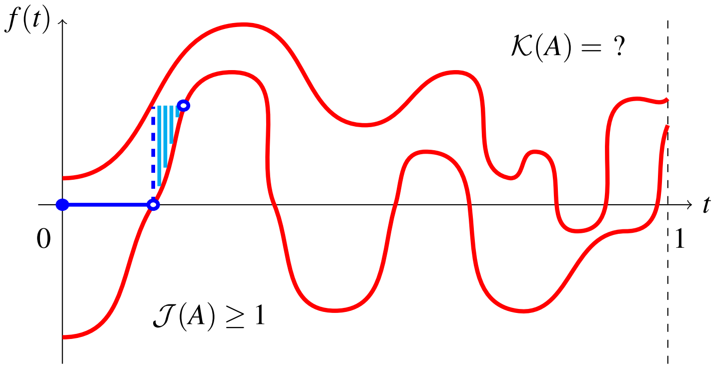

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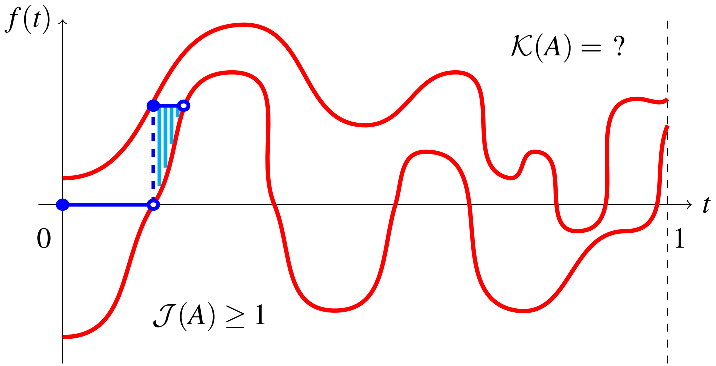

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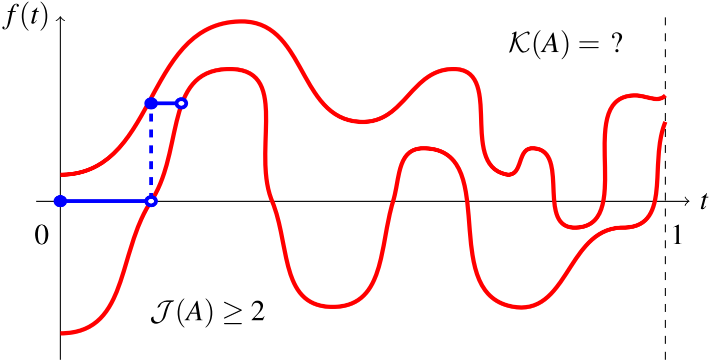

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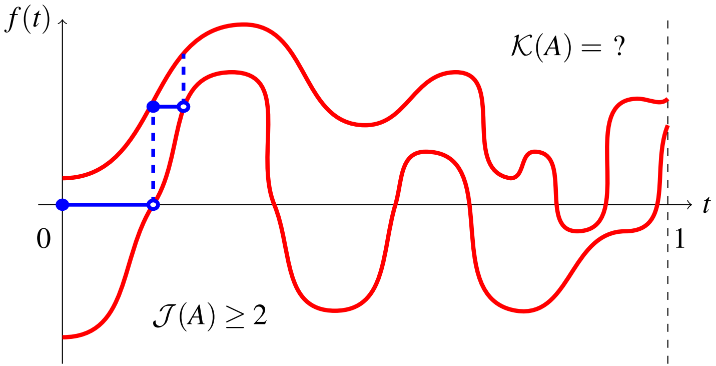

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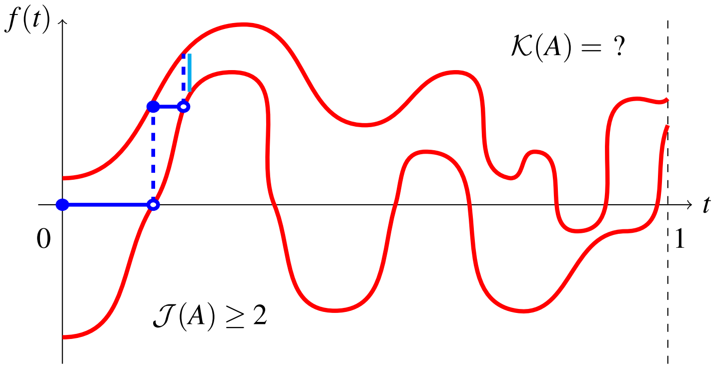
Example: Sausage



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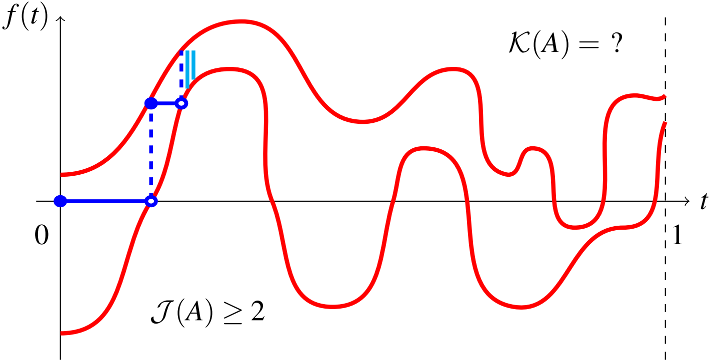

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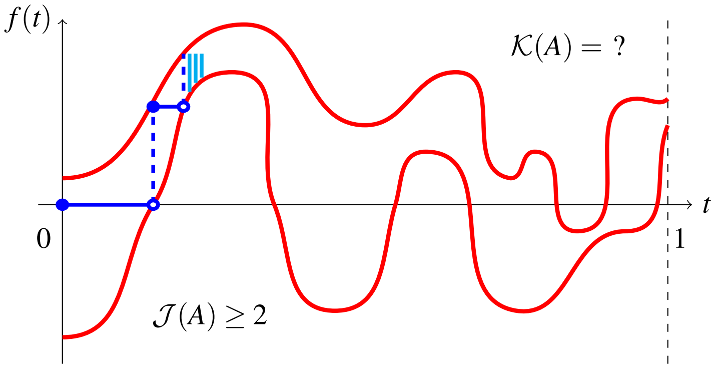

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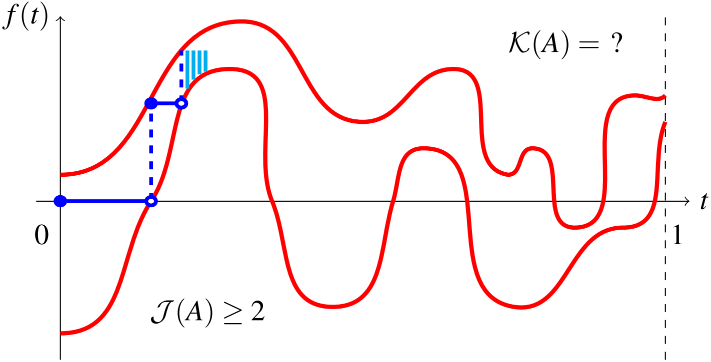

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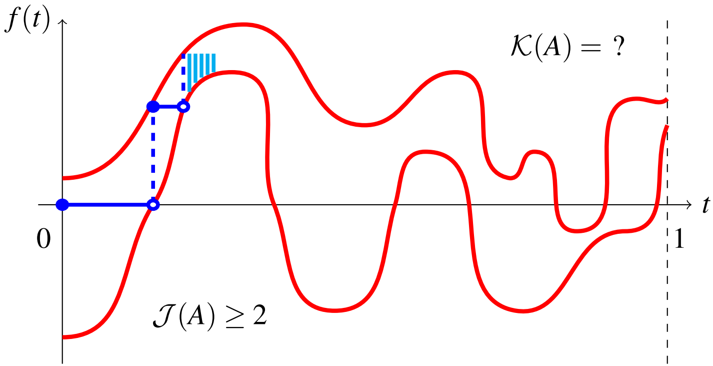

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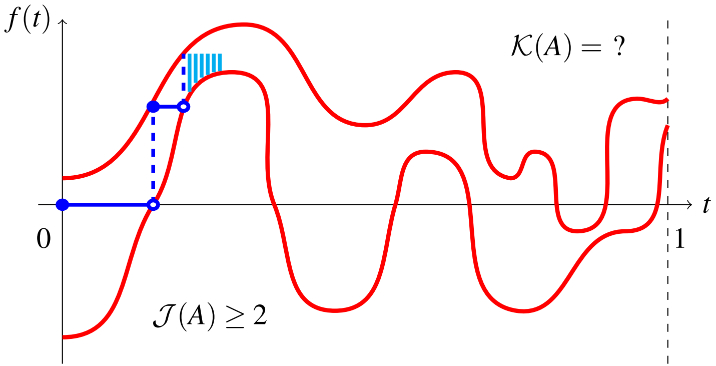

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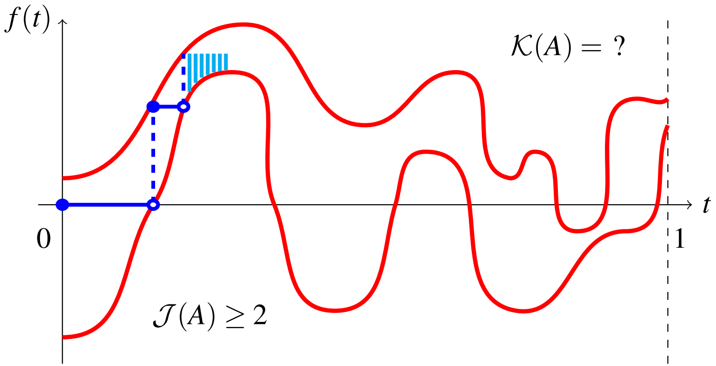

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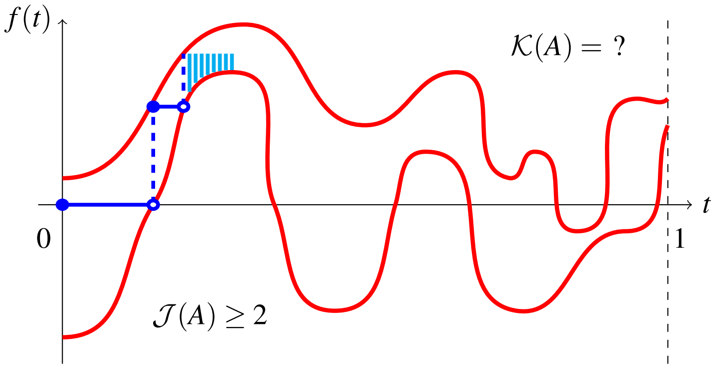

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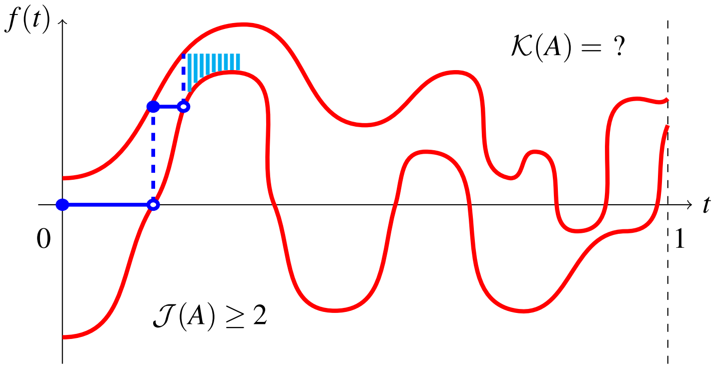

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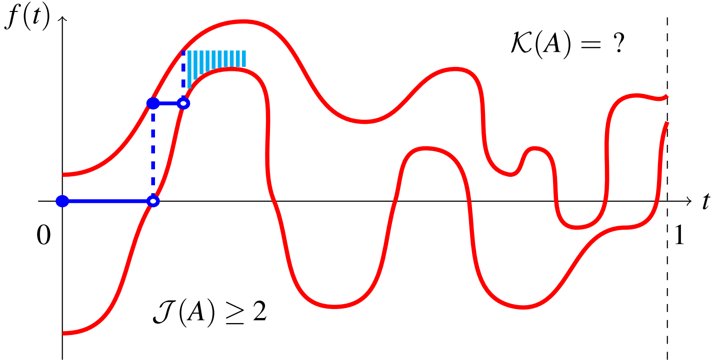

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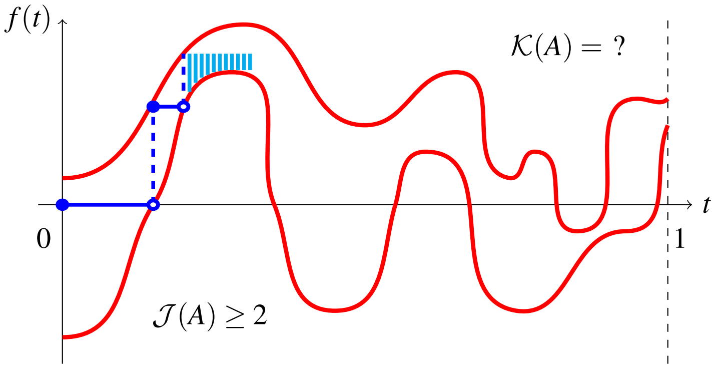

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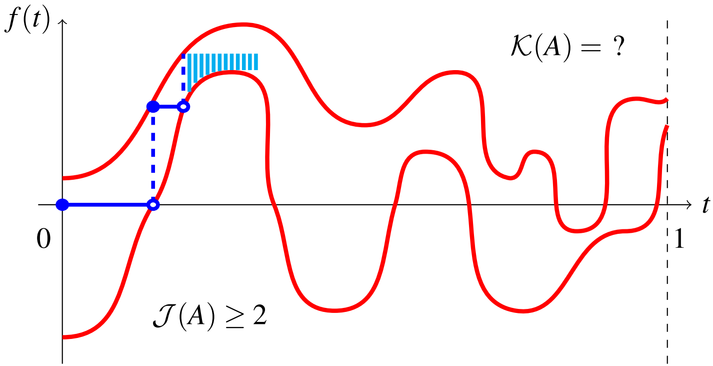

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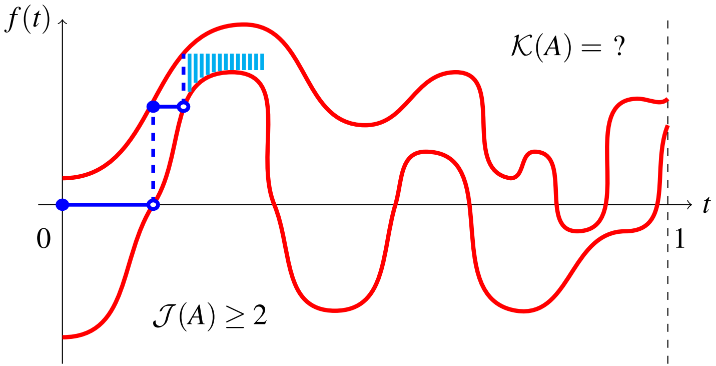

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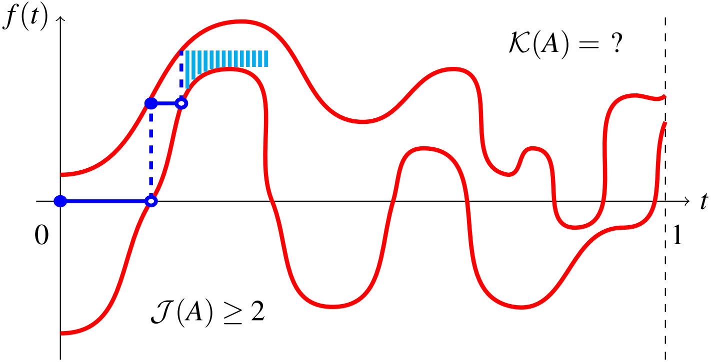

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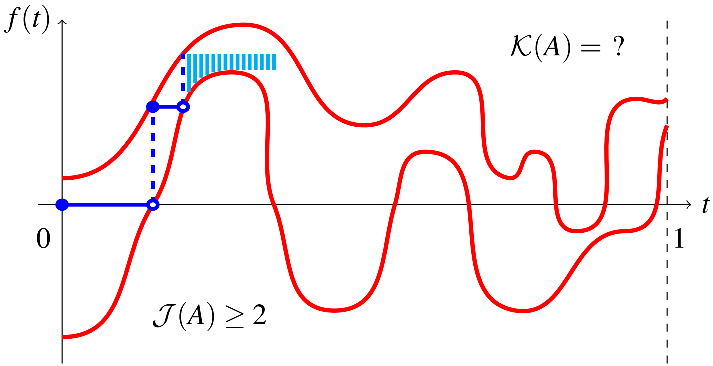

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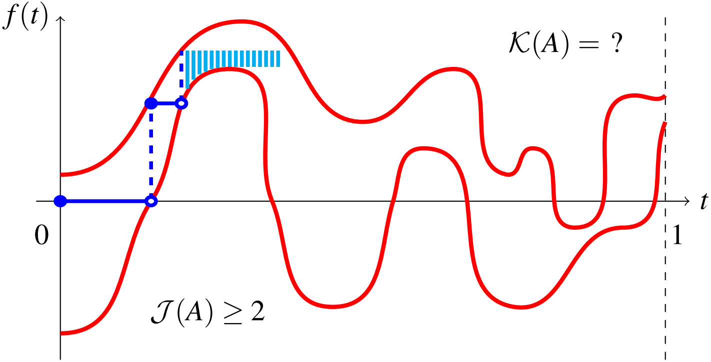

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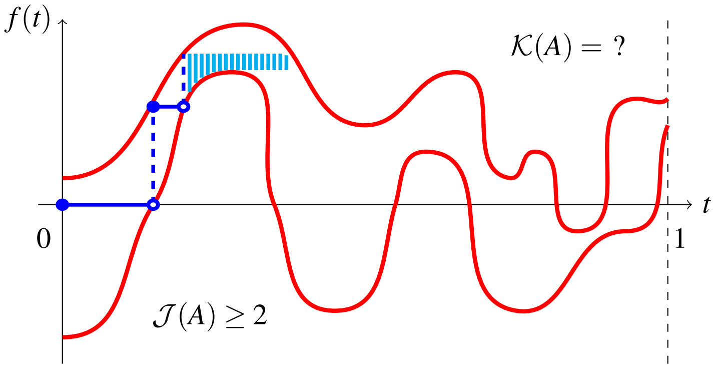

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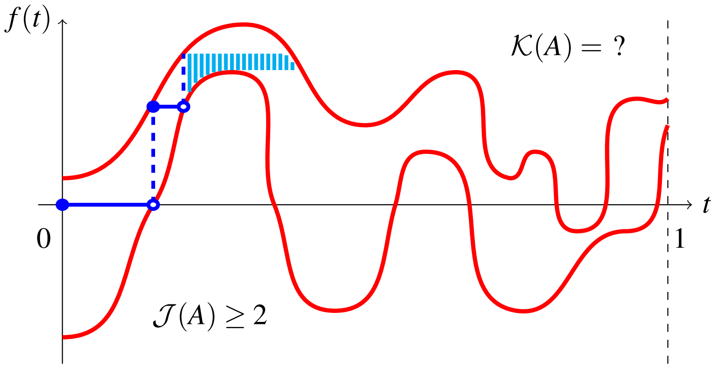

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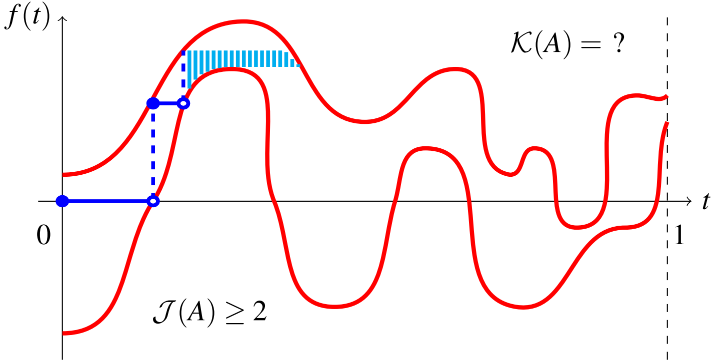

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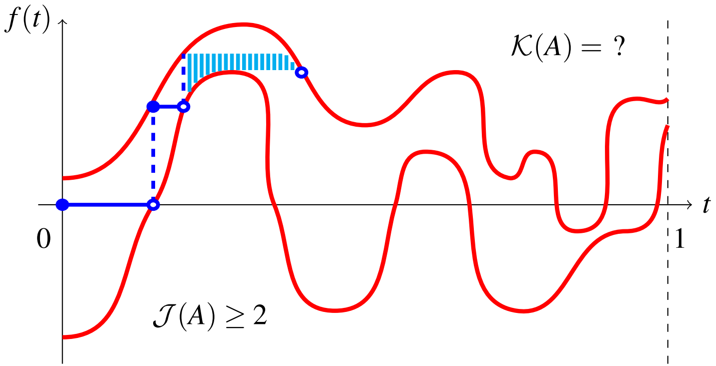

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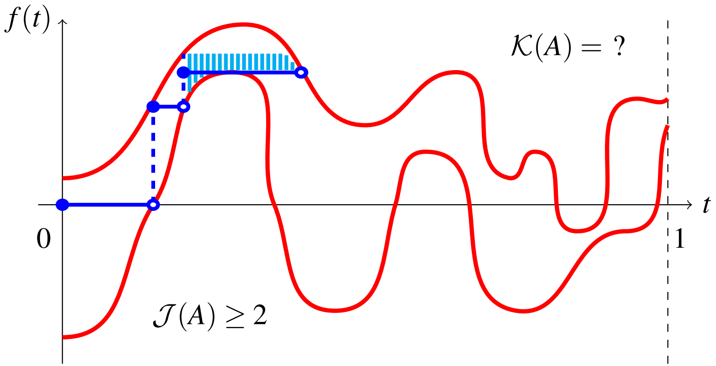

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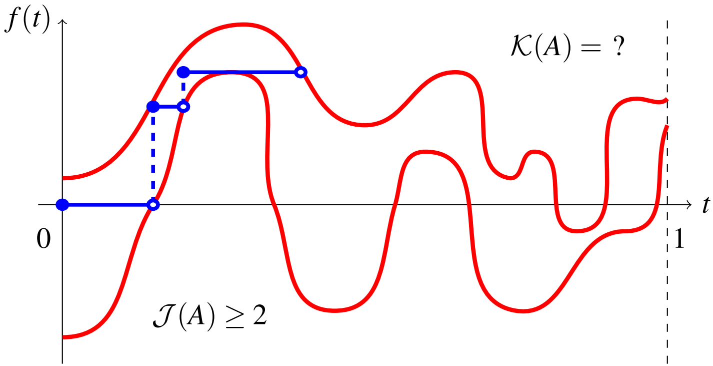

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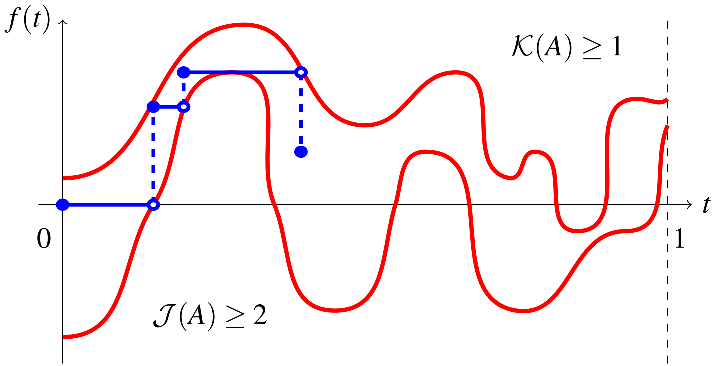

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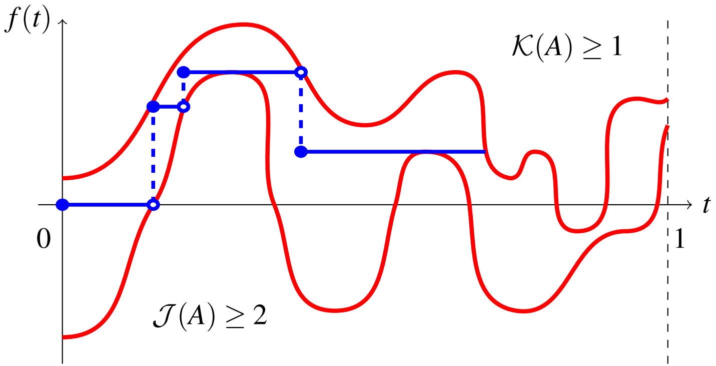

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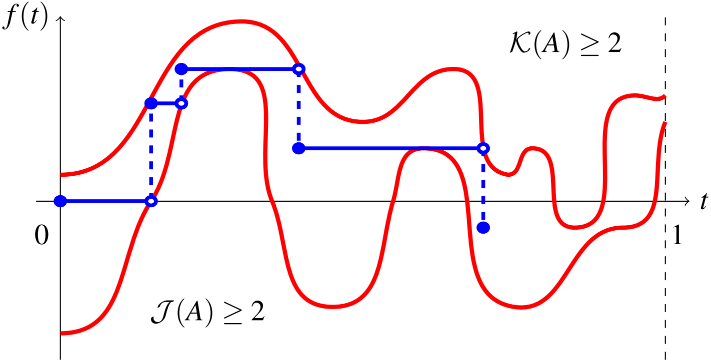

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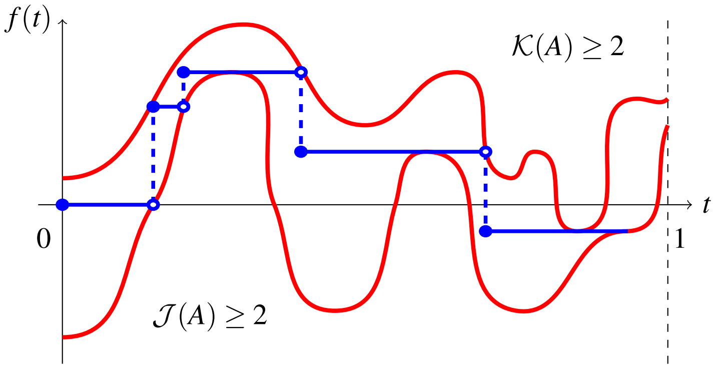

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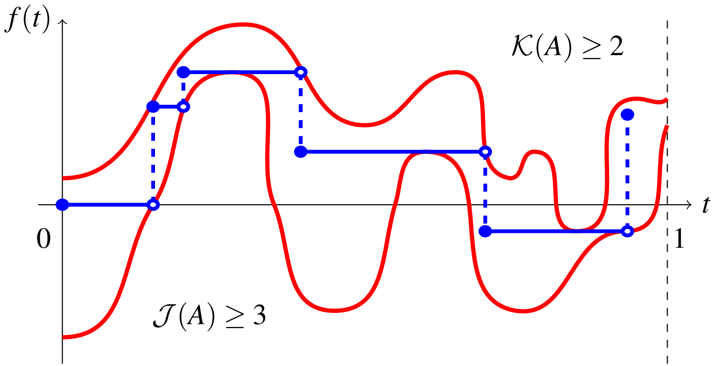

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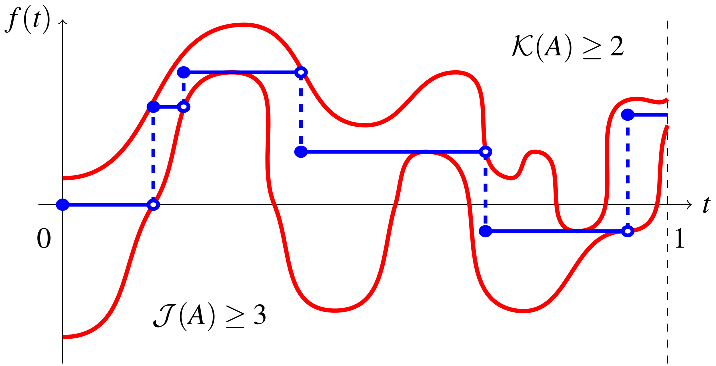

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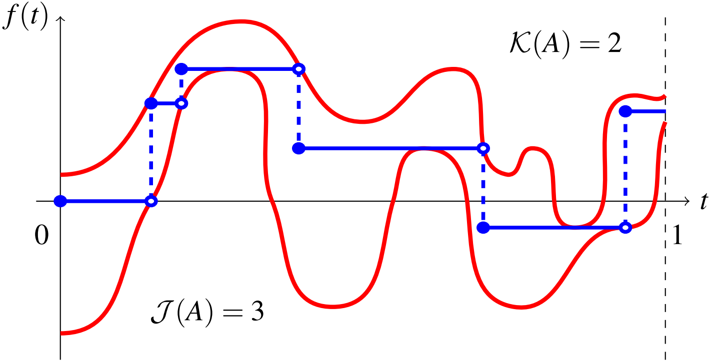

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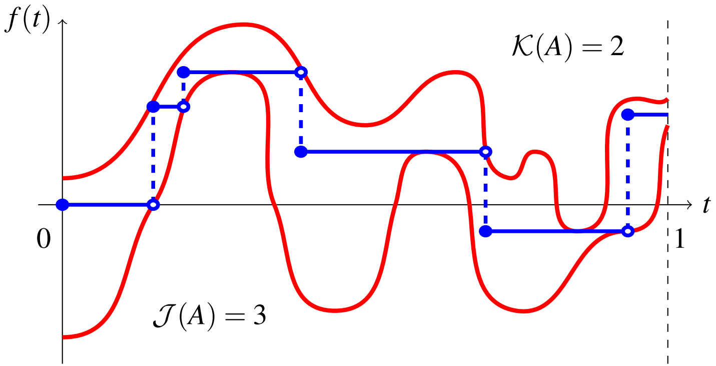

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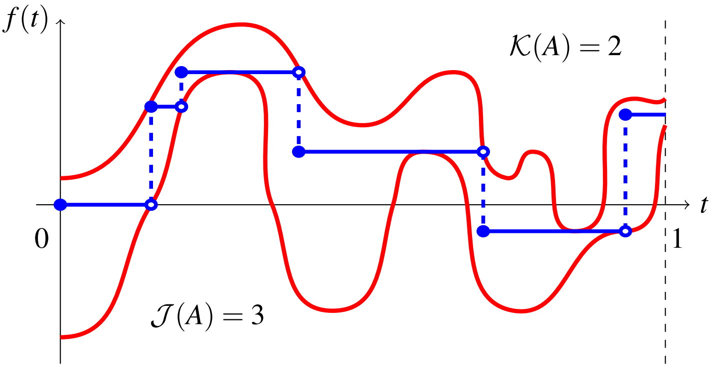

- $A = \{f \in \mathbb{D} : f$ stays between the two curves$\}$
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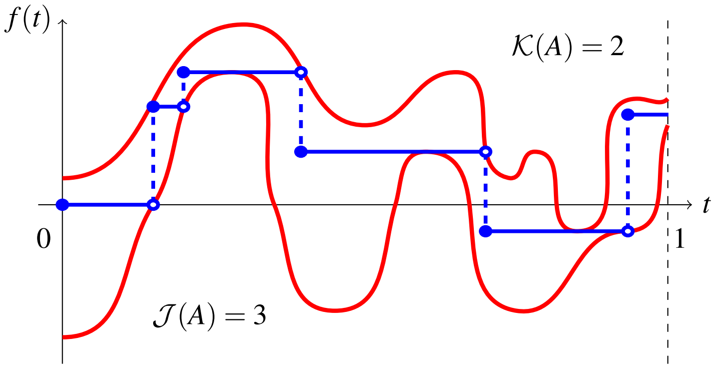

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- $\mathbf{P}(\bar{S}_n \in \cdot | \bar{S}_n \in A) \to \mathbf{P}\Big(\sum_{j=1}^{3} Z_j^{+} \mathbb{1}_{[U_j^{+} \leq t]} - \sum_{k=1}^{2} Z_k^{-} \mathbb{1}_{[U_k^{-} \leq t]} \in \cdot\Big)$

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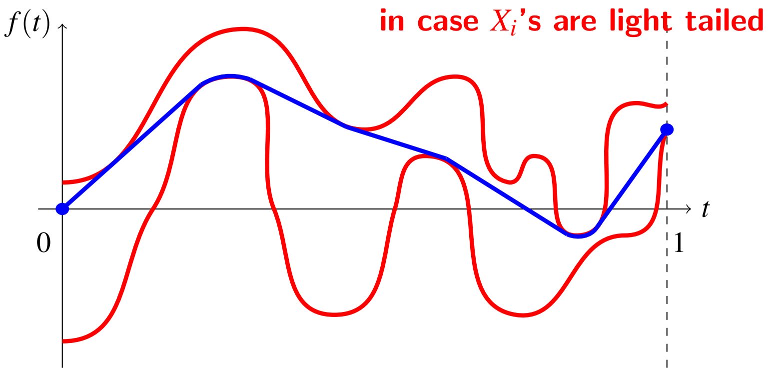

- $A = \{f \in \mathbb{D} : f$ stays between the two curves$\}$
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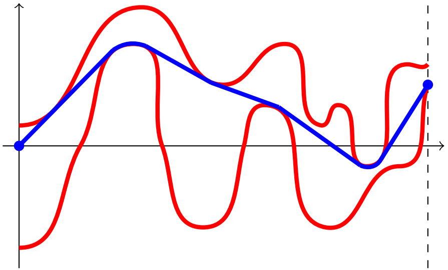
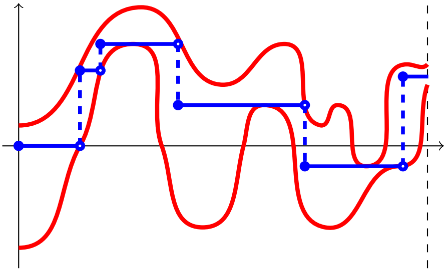
I^*: sol'n of optimization problem over absolutely continuous functions.

Example: Sausage





Conspiracy

Connection to variational problems

(continuous optimization)

Catastrophe

Connection to impulse control

(discrete optimization)

Lévy Processes and Multidimensional Processes

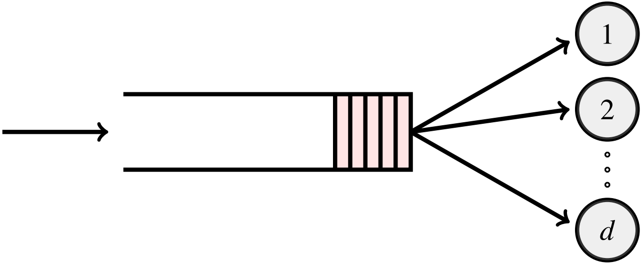
- Same results for Lévy processes.

- Similar results for vector valued processes with independent components as well (for both Lévy processes and random walks).

Analysis of Many Server Queues and even Queueing Networks!

Example: Many-Server Queue

Congestion of Multiple Server Queue:



How many big jobs are needed to create large queue lengths?

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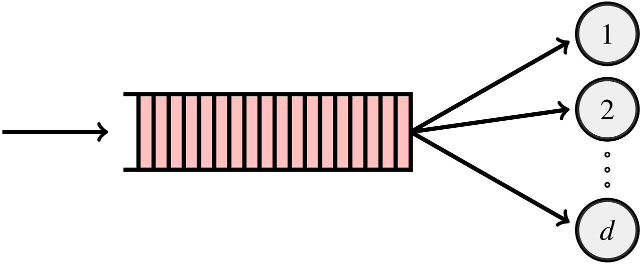

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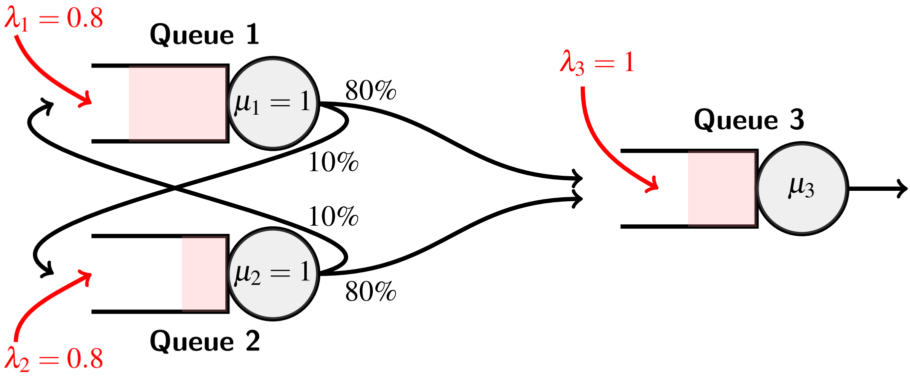
Example: Stochastic Fluid Network



- Job size distribution: Pareto with $\alpha_1 = 1$, $\alpha_2 = 1$, $\alpha_3 = 3$
- Queue 3 experiences congestion because of what?

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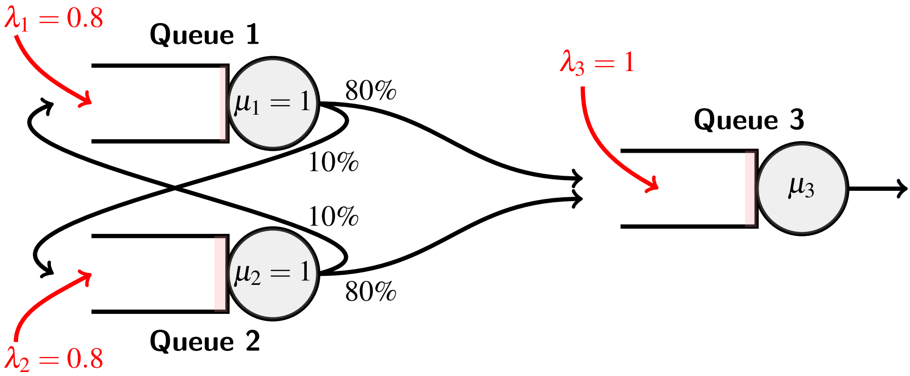

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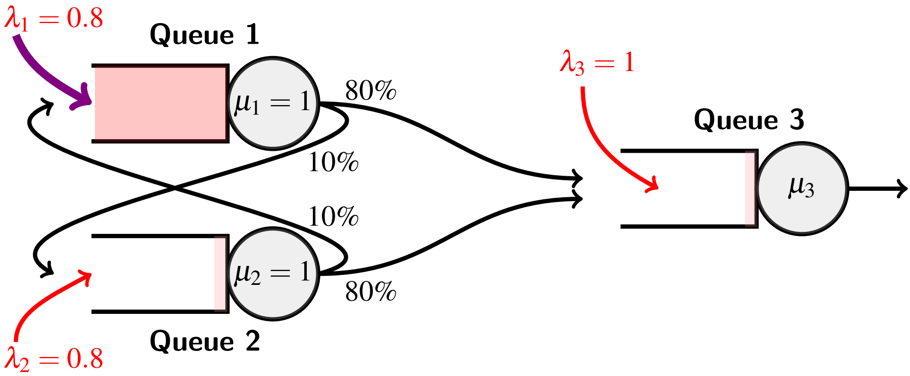

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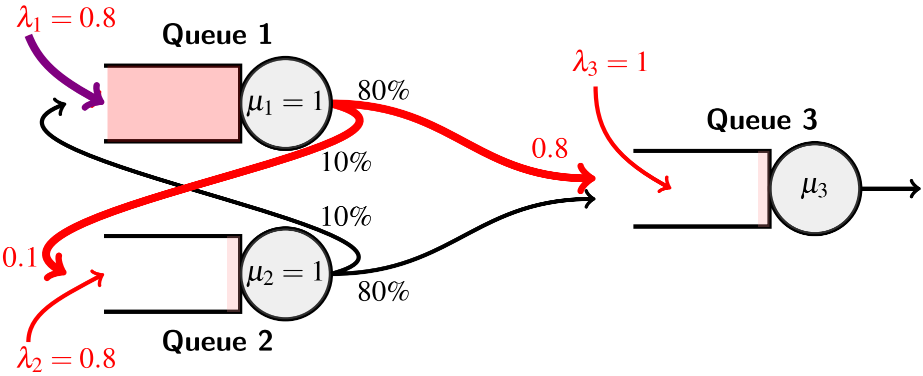

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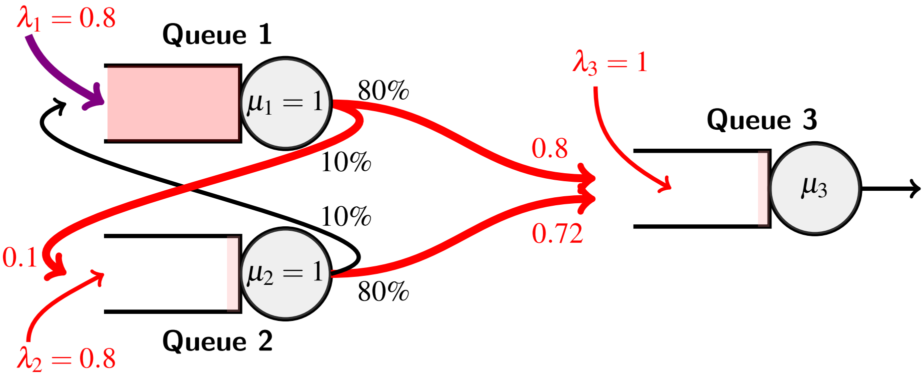

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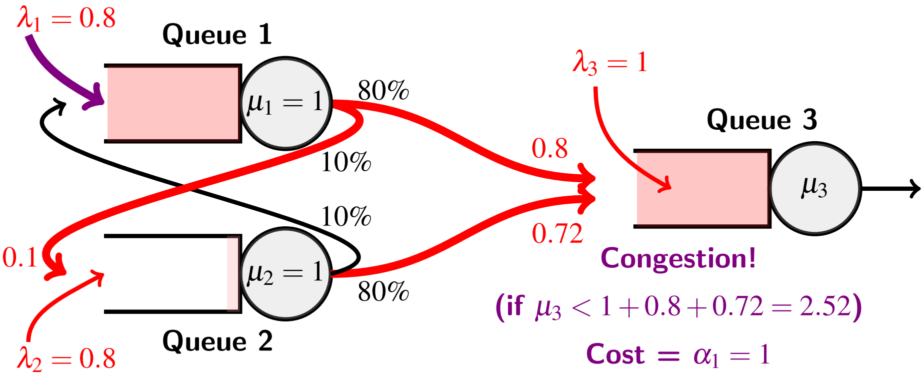

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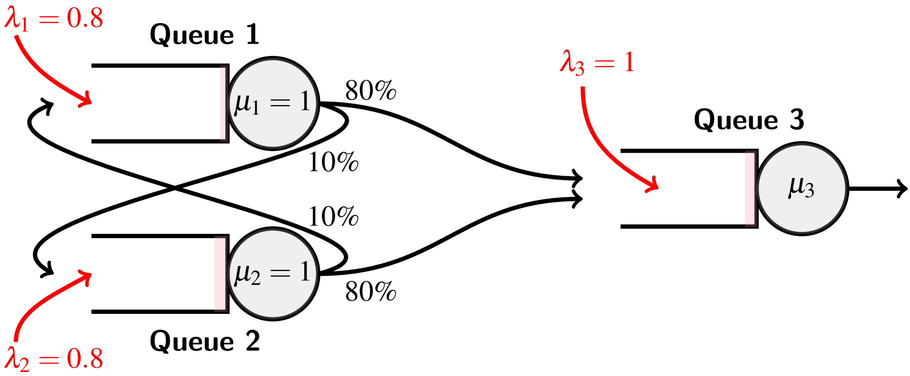

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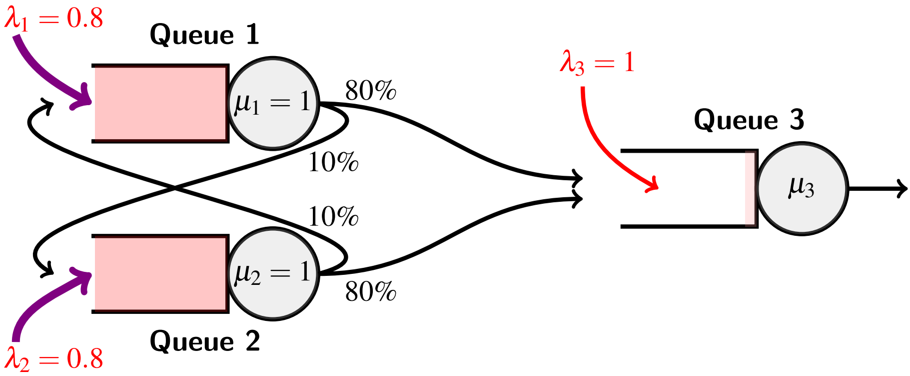

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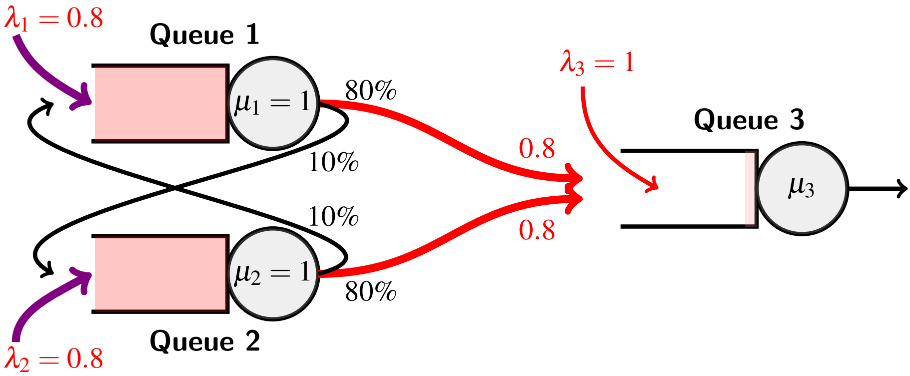

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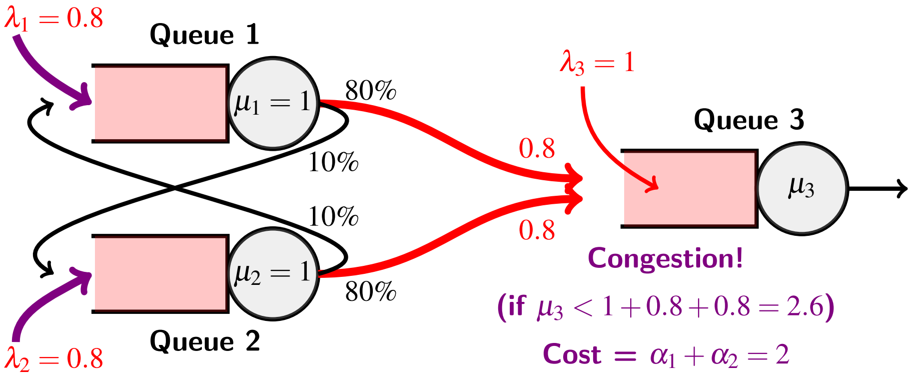

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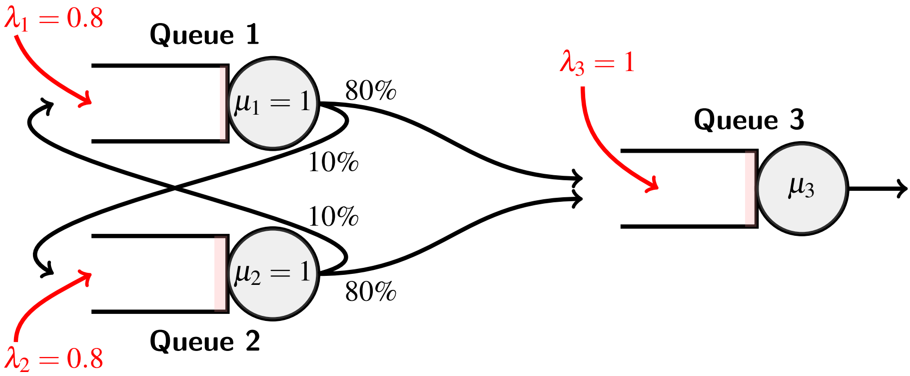

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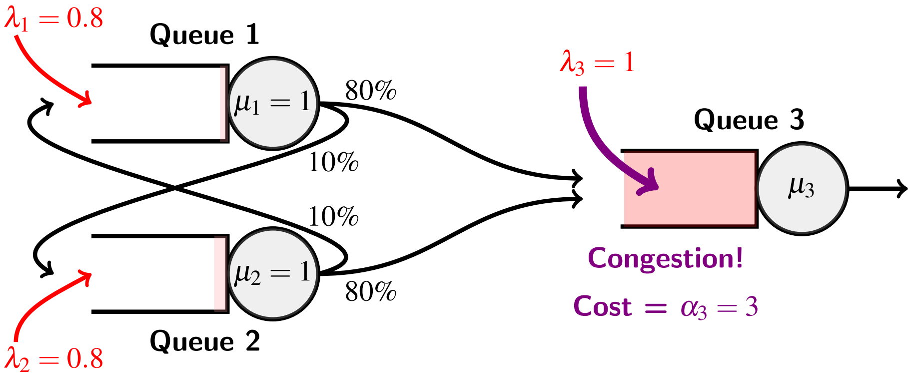

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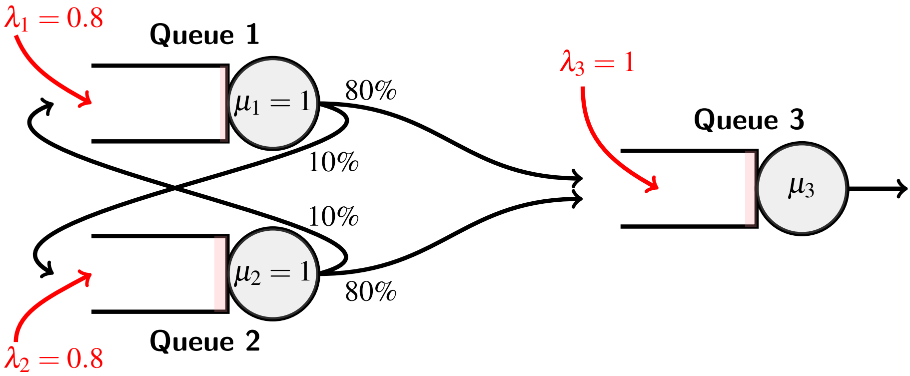

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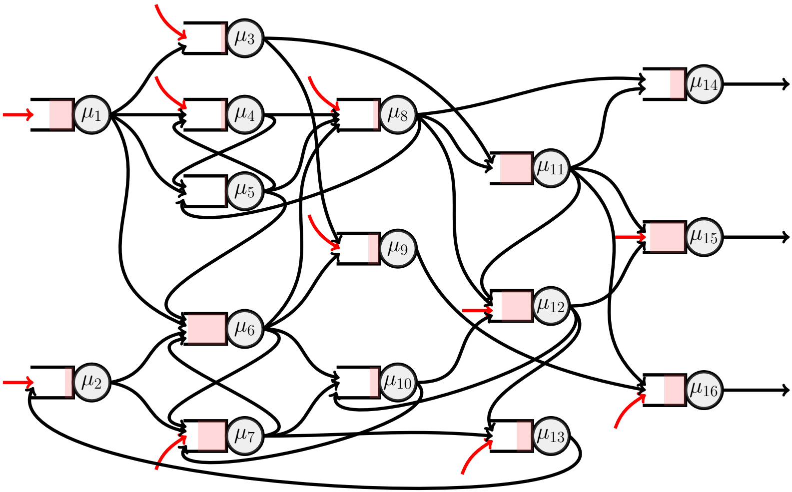
For Larger Network, Knapsack-Type Problem Arises



Another Application: AR(1) process

- $X_{n+1} = A_{n+1} X_n + 1, \qquad X_0 = 0, \qquad A_n \geq 0$ i.i.d.

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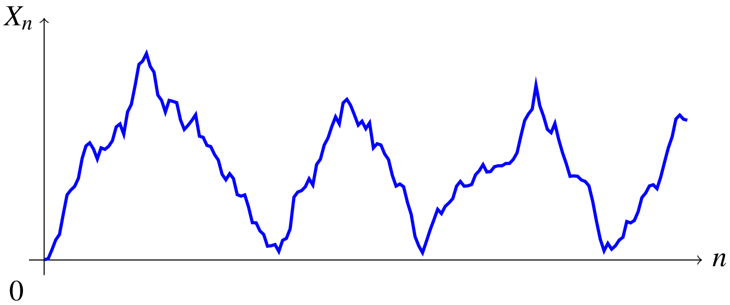
- **Goal:** sample path LD for $\bar{X}_n = \{\bar{X}_n(t), t \in [0,1]\}$

Idea: View Regeneration Cycle as Heavy-Tailed Increment

- Recall $\bar{X}_n(t) \triangleq (X_0 + \cdots + X_{\lfloor nt \rfloor})/n$, where $X_{n+1} = A_{n+1}X_n + 1$



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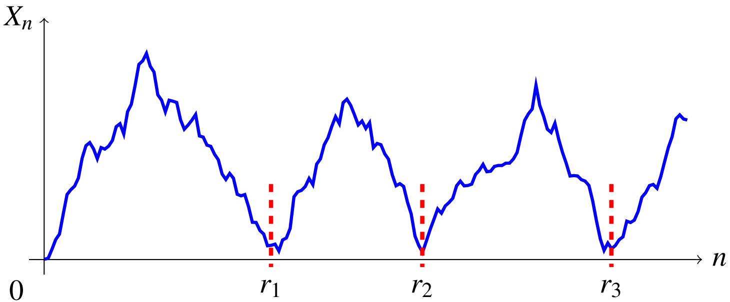

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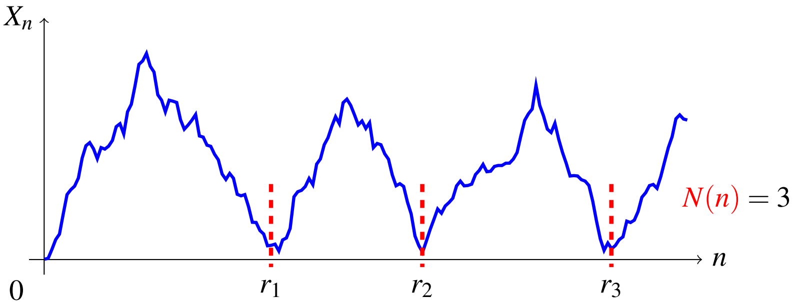

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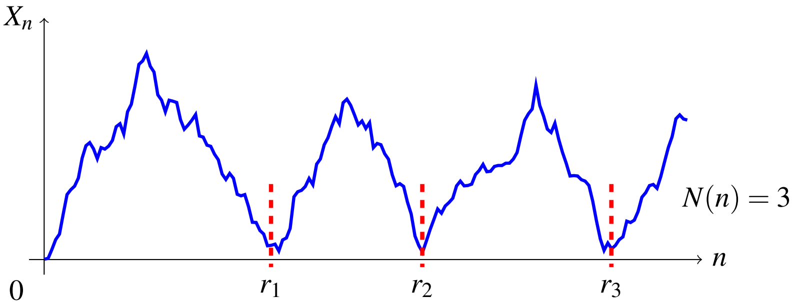
- $\bar{X}_n(t) = (\underbrace{X_0 + \cdots + X_{r_1 - 1}}_{\triangleq X_1'} + X_2' + \cdots + X'_{N(nt)} + X_{r_{N(nt)}} + \cdots + X_{\lfloor nt \rfloor})/n$



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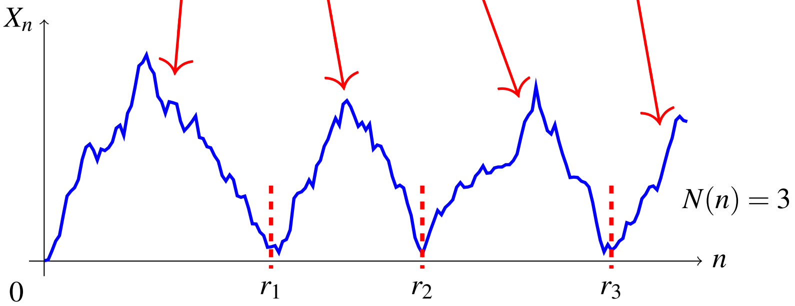

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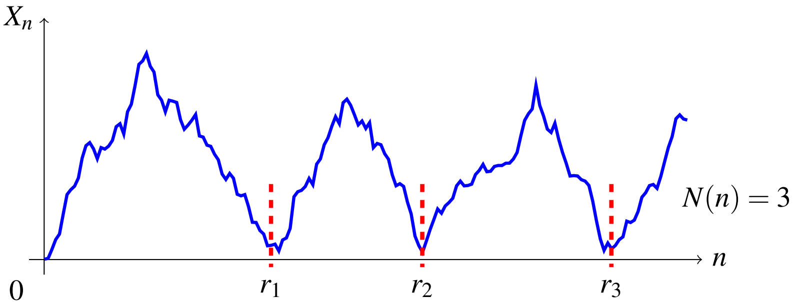
- **First step:** sample path LD for $\bar{X}'_n = \{(X_1' + \cdots + X'_{N(nt)})/n, t \in [0,1]\}$



Tail Asymptotics of X'_i (Area under Regeneration Cycle)

- Recall $\bar{X}'_n = \{(X'_1 + \cdots + X'_{N(nt)})/n, t \in [0,1]\}$,

  $X'_1 = X_0 + \cdots + X_{r_1 - 1}$: the area under regeneration cycle

- $\mathbf{P}(X_\infty > u) \sim C_\infty u^{-\alpha}$ (Recall α is s.t. $\mathbf{E}A_1^\alpha = 1$)

- **Question:** X'_1 asymptotic power law?

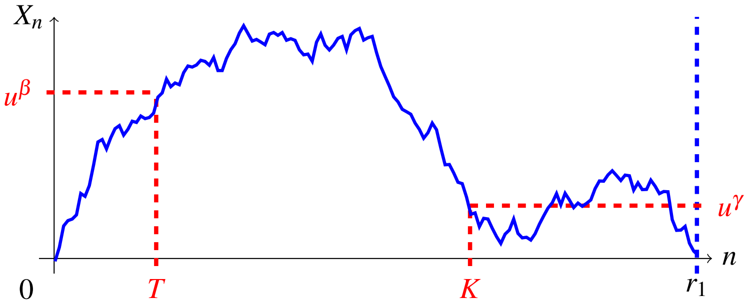
Theorem

Let α be s.t. $\mathbf{E}A_1^\alpha = 1$, then there exists $C > 0$ s.t.

$$\mathbf{P}(X'_1 > u) \sim Cu^{-\alpha}.$$

Ideas behind the Proof of $\mathbf{P}(X_1' > u) \sim Cu^{-\alpha}$

- $T \triangleq \inf\{n : X_n > u^\beta\}, \quad K \triangleq \inf\{n > T : X_n \le u^\gamma\}, \quad 0 < \gamma < \beta < 1$



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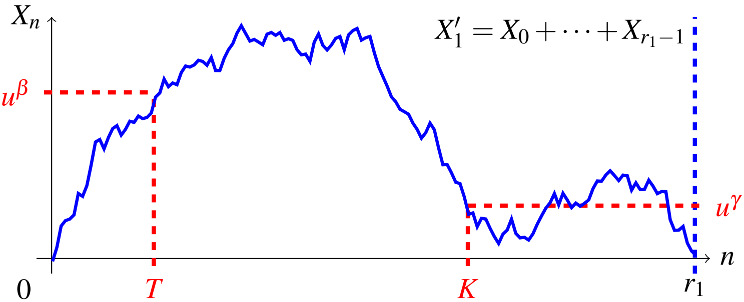

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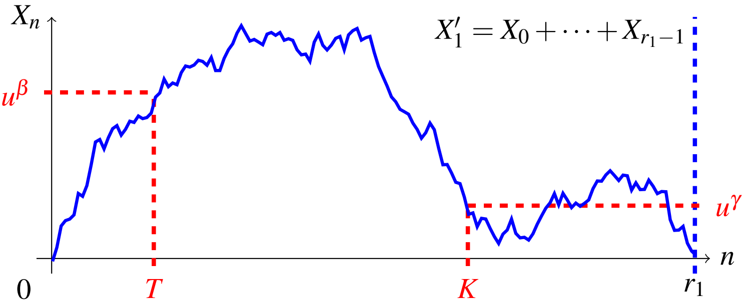

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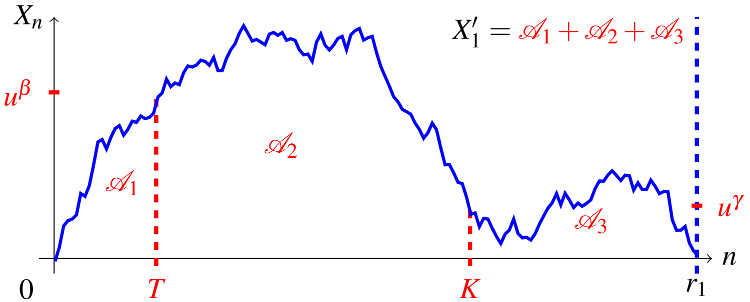

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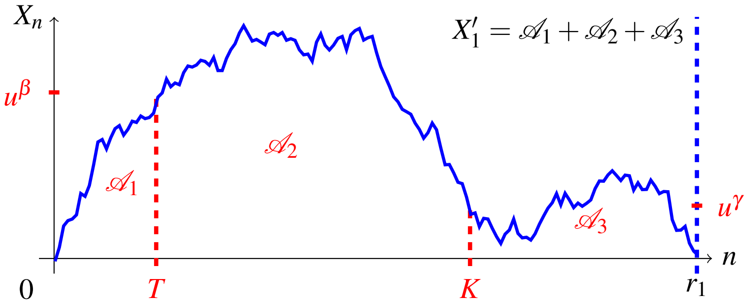

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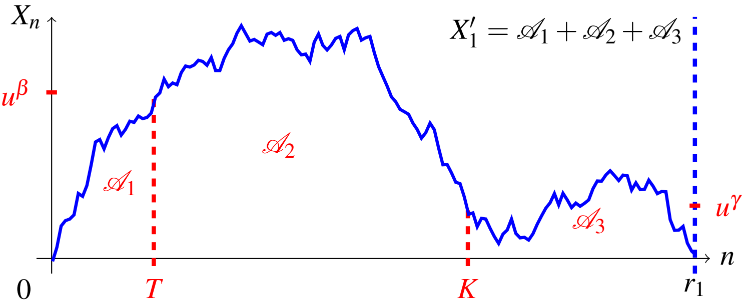

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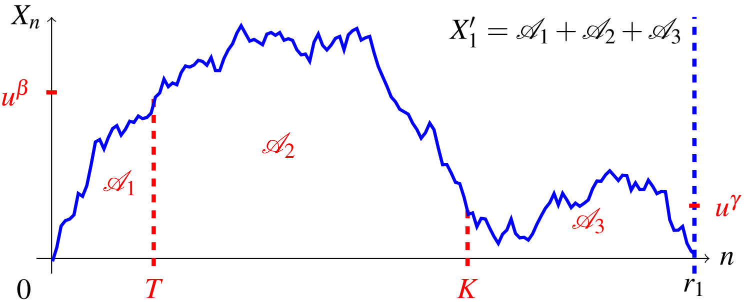

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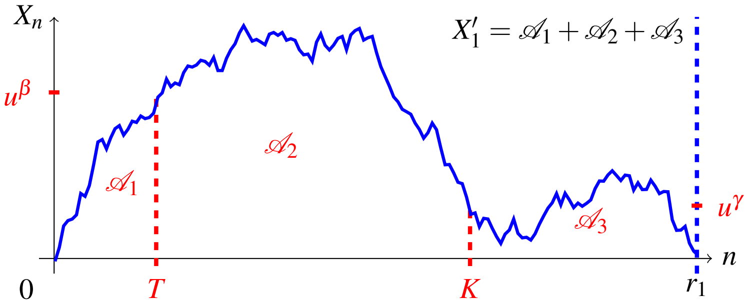

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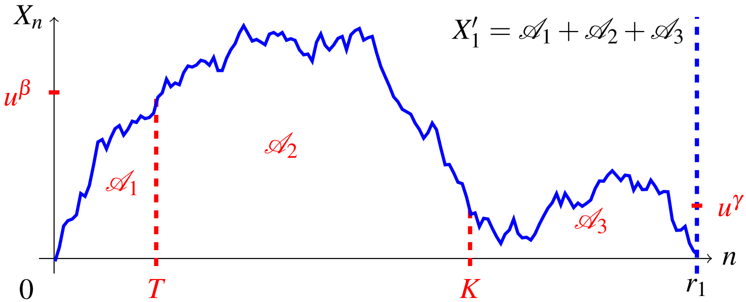

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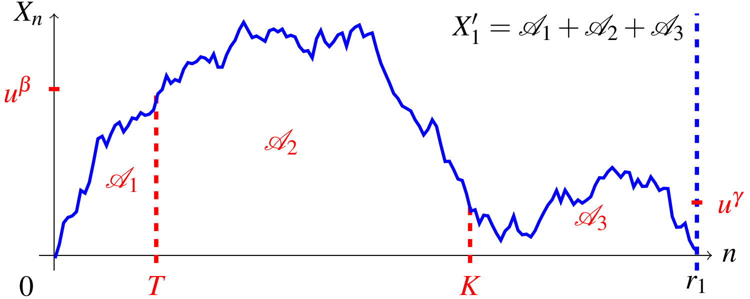

Part 2. Heavy-Tailed Rare Event Simulation

Computing $\mathbf{P}(\bar{S}_n \in A)$ for finite n

Challenge of Rare Event Simulation

Monte Carlo simulations as repetitive random experiments:

e.g. Coin flip: want to estimate $\mathbf{P}(\text{Head})$

- Flip the coin 100 times
- Count the number of head
- Divide by 100 and report the number

Should be reasonably close to $1/2$

Challenge of Rare Event Simulation

Monte Carlo simulations as repetitive random experiments:

e.g. Coin flip: want to estimate $\mathbf{P}(\text{Edge})$

- Flip the coin 100 times
- Count the number of Edge
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Challenge of Rare Event Simulation

Monte Carlo simulations as repetitive random experiments:

e.g. Coin flip: want to estimate $\mathbf{P}(\text{Edge})$

- Flip the coin 100 times
- Count the number of Edge : most likely to be 0
- Divide by 100 and report the number

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Monte Carlo simulations as repetitive random experiments:

e.g. Coin flip: want to estimate $\mathbf{P}(\text{Edge})$

- Flip the coin 100 times
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- Divide by 100 and report the number : and hence, likely to be 0

Challenge of Rare Event Simulation

Monte Carlo simulations as repetitive random experiments:

e.g. Coin flip: want to estimate $\mathbf{P}(\text{Edge})$

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- Count the number of Edge : most likely to be 0
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Is 0 a useful answer?

Challenge of Rare Event Simulation

Monte Carlo simulations as repetitive random experiments:

e.g. Coin flip: want to estimate $\mathbf{P}$(Edge)

- Flip the coin 100 times
- Count the number of Edge : most likely to be 0
- Divide by 100 and report the number : and hence, likely to be 0

Is 0 a useful answer? No.

e.g., Nuclear Meltdown, Large-Scale Blackout, Large Financial Loss

Challenge of Rare Event Simulation

Monte Carlo simulations as repetitive random experiments:

e.g. Coin flip: want to estimate $\mathbf{P}(\text{Edge}) \overset{\textit{Suppose}}{\approx} 10^{-6}$

- Flip the coin 100 times
- Count the number of Edge
- Divide by 100 and report the number

Challenge of Rare Event Simulation

Monte Carlo simulations as repetitive random experiments:

e.g. Coin flip: want to estimate $\mathbf{P}(\text{Edge}) \overset{\text{Suppose}}{\approx} 10^{-6}$

- Flip the coin ~~100~~ a few million times
- Count the number of Edge
- Divide by the total number of flips and report the number

Challenge of Rare Event Simulation

Monte Carlo simulations as repetitive random experiments:

e.g. Coin flip: want to estimate $\mathbf{P}(\text{Edge}) \overset{Suppose}{\approx} 10^{-6}$

- Flip the coin ~~100~~ a few million times
- Count the number of Edge
- Divide by the total number of flips and report the number

Much harder than $\mathbf{P}$(Head)

Importance Sampling

- Construct an alternative universe

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- Recover the true probability using the relationship between the two parallel universes

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- Recover the true probability using the relationship between the two parallel universes
  - i.e., Report $\frac{1}{m}\sum_{i=1}^{m} \mathbb{I}_{\mathsf{Edge}}^{(i)} \left(\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}\mathbf{Q}}\right)^{(i)}$ as an estimate of $\mathbf{P}(\mathsf{Edge})$

Importance Sampling for $\mathbf{P}(\bar{S}_n \in A)$

- Construct an alternative universe
  - i.e., Consider an "importance distribution" $\mathbf{Q}$

- Perform experiments there
  - i.e., Sample $\mathbb{I}^{(1)}_{\text{Edge}}, \dots, \mathbb{I}^{(m)}_{\text{Edge}}$ from $\mathbf{Q}$

    as well as the likelihood ratio $\left(\frac{d\mathbf{P}}{d\mathbf{Q}}\right)^{(1)}, \dots, \left(\frac{d\mathbf{P}}{d\mathbf{Q}}\right)^{(m)}$

- Recover the true probability using the relationship between the two parallel universes
  - i.e., Report $\frac{1}{m}\sum_{i=1}^{m} \mathbb{I}^{(i)}_{\text{Edge}} \left(\frac{d\mathbf{P}}{d\mathbf{Q}}\right)^{(i)}$ as an estimate of $\mathbf{P}(\text{Edge})$

Importance Sampling for $\mathbf{P}(\bar{S}_n \in A)$

- Construct an alternative universe
  - i.e., Consider an "importance distribution" $\mathbf{Q}_n$

- Perform experiments there
  - i.e., Sample $\mathbb{I}^{(1)}_{\{\bar{S}_n \in A\}}, \ldots, \mathbb{I}^{(m)}_{\{\bar{S}_n \in A\}}$ from $\mathbf{Q}_n$

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Finding a good alternative universe $\mathbf{Q}_n$ is crucial.

What is a good alternate universe $\mathbf{Q}_n$ for $\mathbf{P}(\bar{S}_n \in A)$?

General principle for making $\overbrace{\mathbb{I}_{\{\bar{S}_n \in A\}} \frac{\mathrm{d}\mathbf{P}}{\mathrm{d}\mathbf{Q}_n}}^{\text{IS estimator}}$ an efficient estimator:

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- **Make sure that $\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}\mathbf{Q}_n}$ does not blow up.**

Goal: Strongly Efficient IS Estimator

$\overbrace{\mathbb{I}_{\{\bar{S}_n \in A\}} \frac{\mathrm{d}\mathbf{P}}{\mathrm{d}\mathbf{Q}_n}}^{\text{IS estimator}}$ is a **strongly efficient** estimator for $\mathbf{P}(\bar{S}_n \in A)$, if

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Notoriously Hard for Heavy-Tailed Processes.

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- Fix $w \in (0,1)$ and define

$$\mathbf{Q}_n(\cdot) \triangleq w\mathbf{P}(\cdot) + (1-w)\mathbf{P}(\cdot \,|\, \bar{S}_n \in B^{\gamma})$$

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$Z_n \triangleq \mathbb{1}_{\{\bar{S}_n \in A\}}\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}\mathbf{Q}_n}$ **is strongly efficient for $\mathbf{P}(\bar{S}_n \in A)$!**

Chen, Blanchet, **R.**, and Zwart (2018)

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$Z_n \triangleq \mathbb{1}_{\{\bar{S}_n \in A\}}\frac{d\mathbf{P}}{d\mathbf{Q}_n}$ **is strongly efficient for $\mathbf{P}(\bar{S}_n \in A)$!**

Chen, Blanchet, **R.**, and Zwart (2018)

First Universal Simulation Scheme for Heavy-Tails

- $B^\gamma \triangleq \{$paths w/ at least $\mathcal{J}(A)$ jumps of size $> \gamma\} \;\Rightarrow\; \mathcal{J}(A) = \mathcal{J}(B^\gamma)$

- Fix $w \in (0,1)$ and define

$$\mathbf{Q}_n(\,\cdot\,) \triangleq w\mathbf{P}(\,\cdot\,) + (1-w)\mathbf{P}(\,\cdot\,|\,\bar{S}_n \in B^\gamma)$$

- If we choose γ so that $\mathcal{J}(A \setminus B^\gamma) \geq \mathcal{J}(A) + \mathcal{J}(B^\gamma) = 2\mathcal{J}(A)$

$$\mathbf{E}^{\mathbf{Q}_n}\left(\mathbb{1}_{\{\bar{S}_n \in A\}} \frac{\mathrm{d}\mathbf{P}}{\mathrm{d}\mathbf{Q}_n}\right)^2 \leq \frac{1}{w}\mathbf{P}(\bar{S}_n \in A \setminus B^\gamma) + \mathbf{P}(\bar{S}_n \in A) \cdot \mathbf{P}(\bar{S}_n \in B^\gamma)$$

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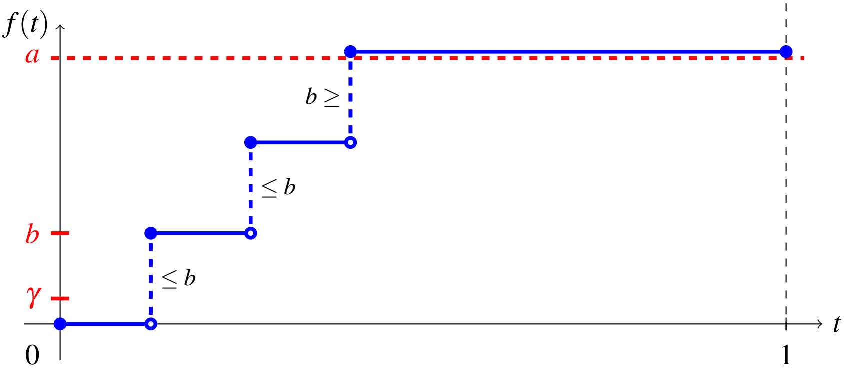
How to ensure $\mathcal{J}(A \setminus B^\gamma) \geq 2\mathcal{J}(A)$: e.g., Reinsurance



- $A \qquad = \{$paths that cross level a on $[0,1]$ & jump sizes $\leq b\}$
- $B^\gamma \qquad = \{$paths with at least 3 jumps of size $> \gamma\}$
- $A \setminus B^\gamma = \{$paths that cross level a on $[0,1]$
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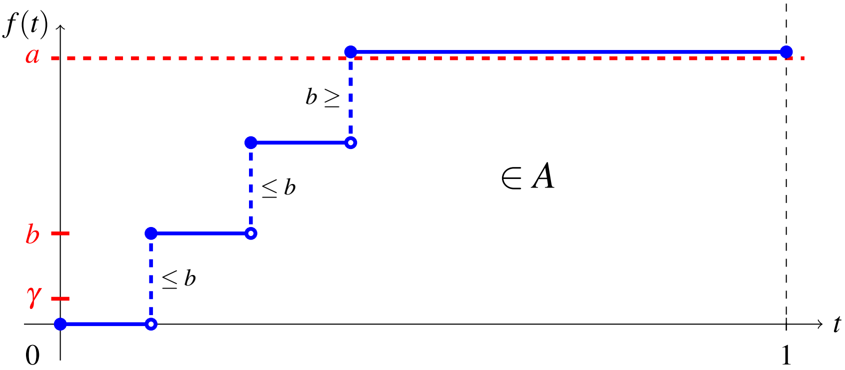

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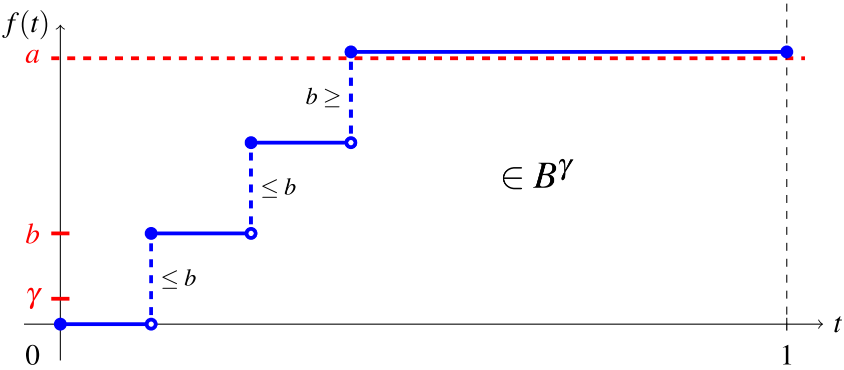

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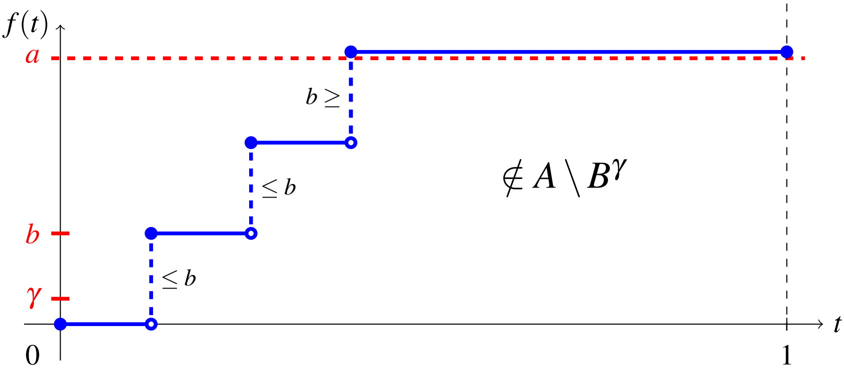

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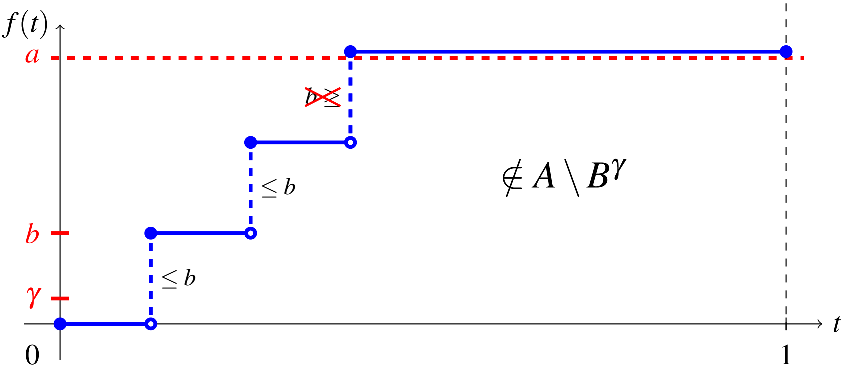

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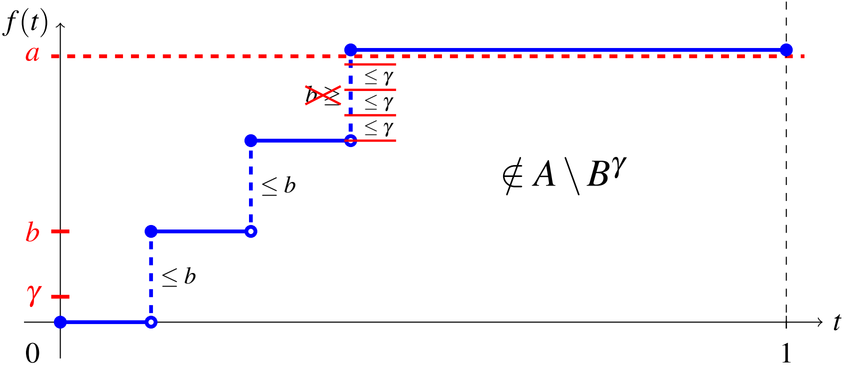

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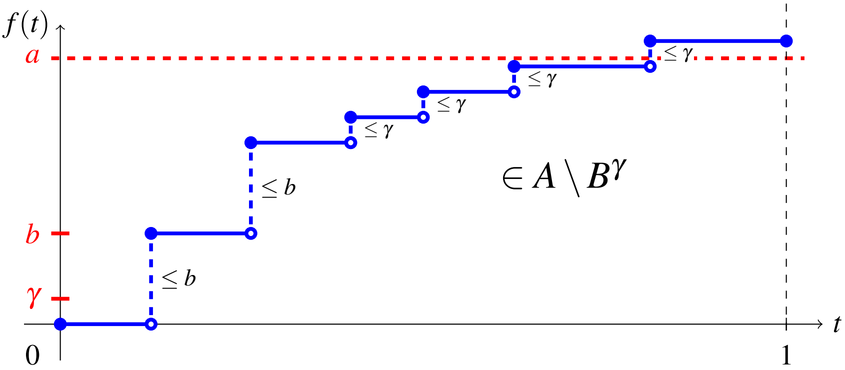

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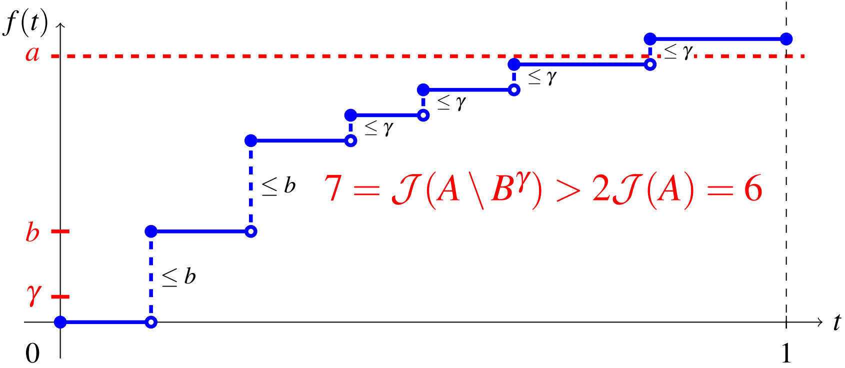

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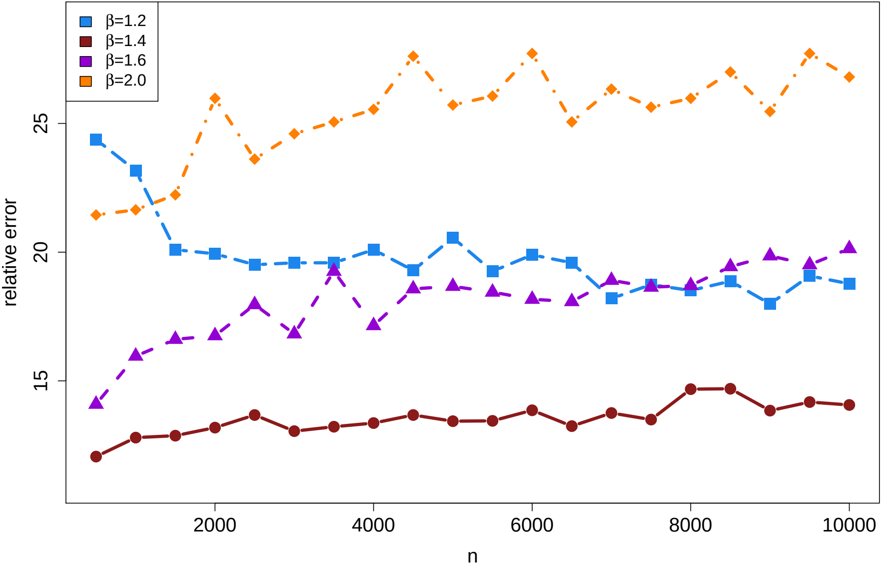
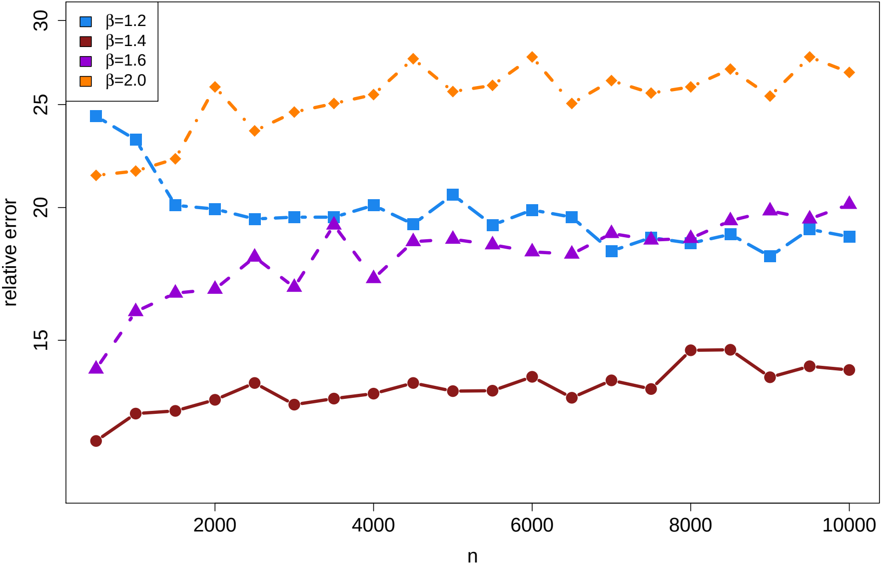
Numerical Experiments for Reinsurance Example



Numerical Results for Reinsurance Example



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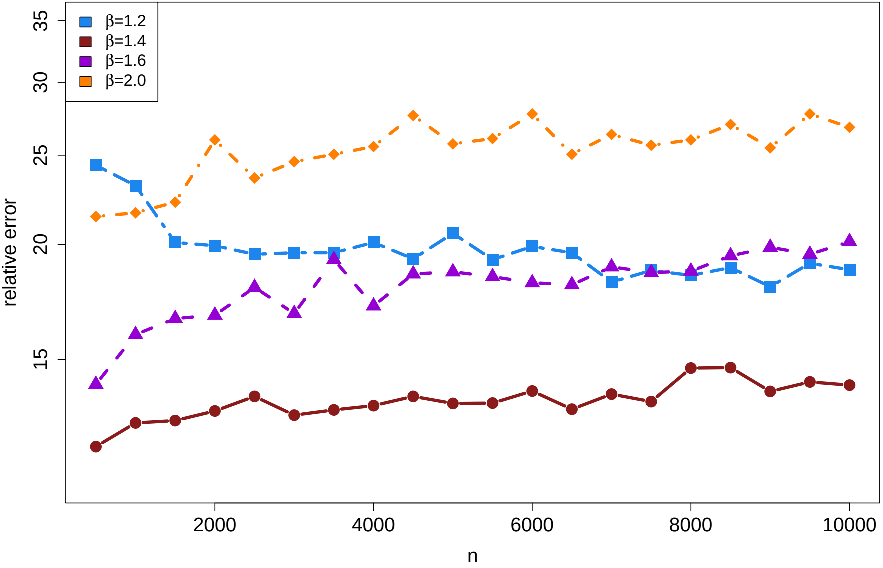

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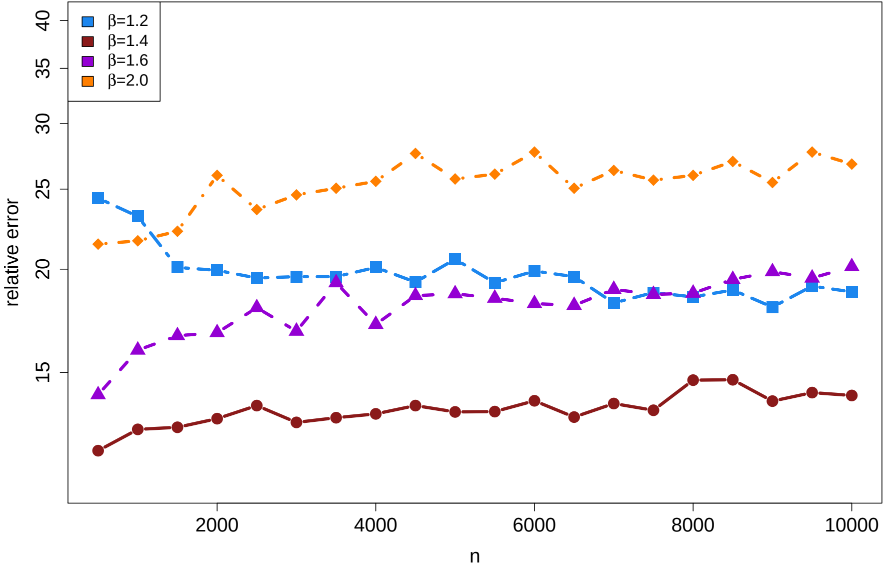

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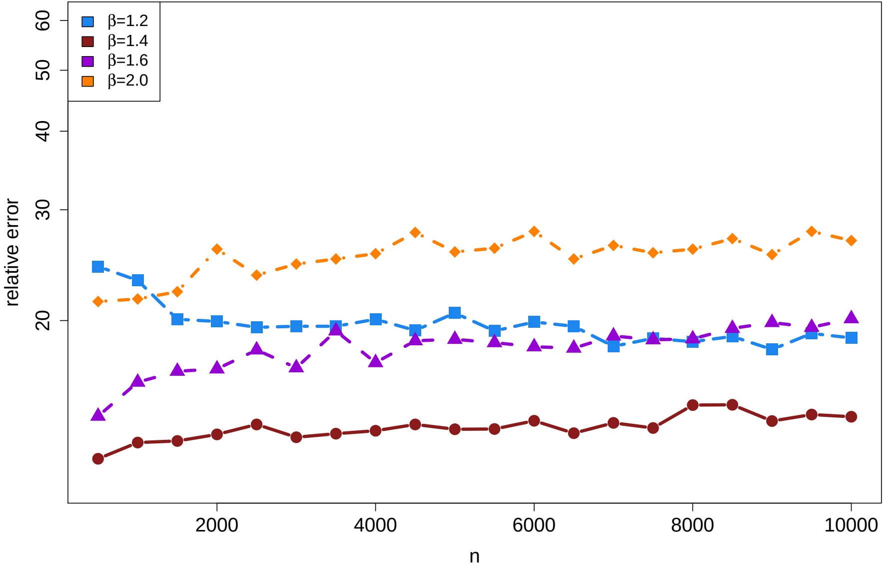

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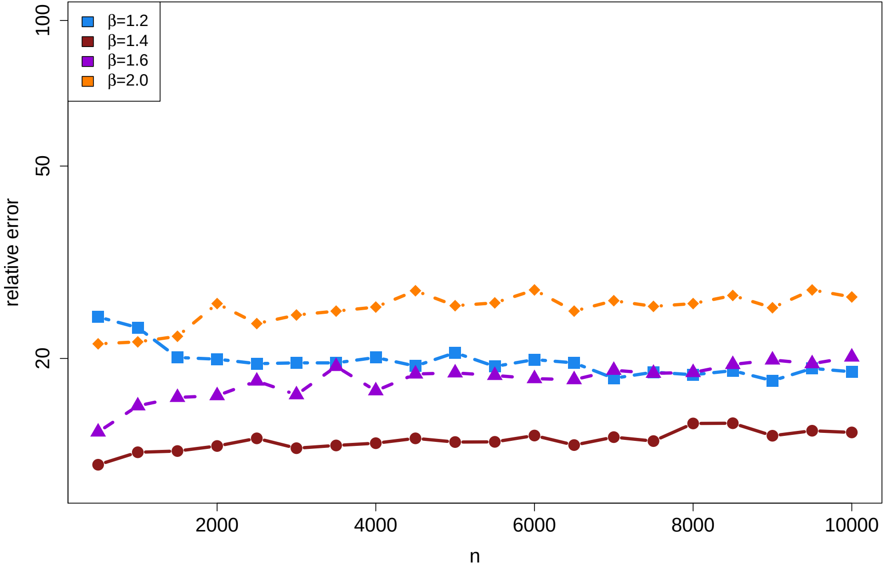

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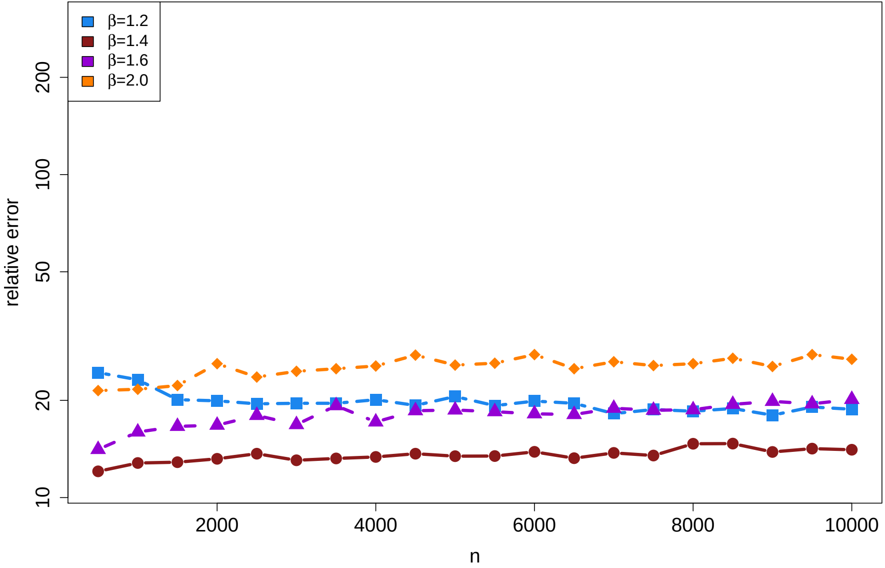

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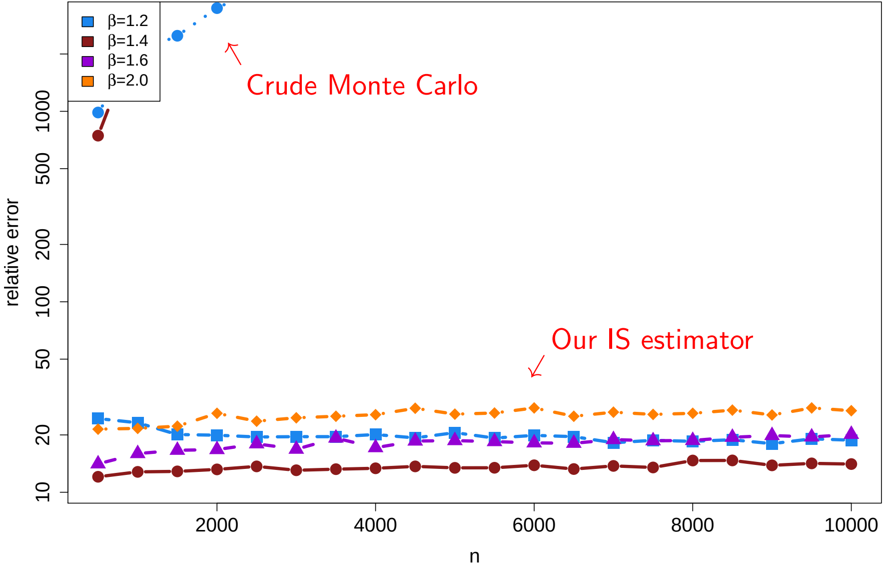

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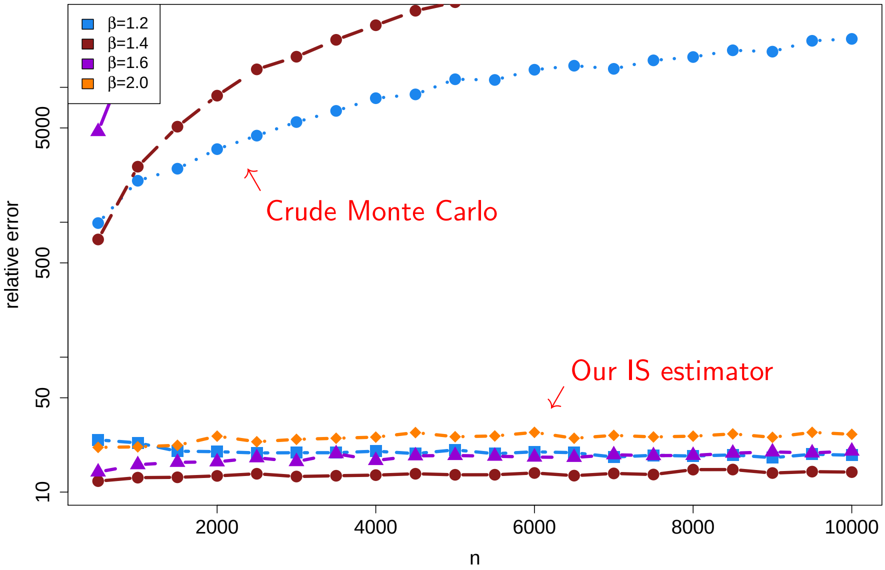

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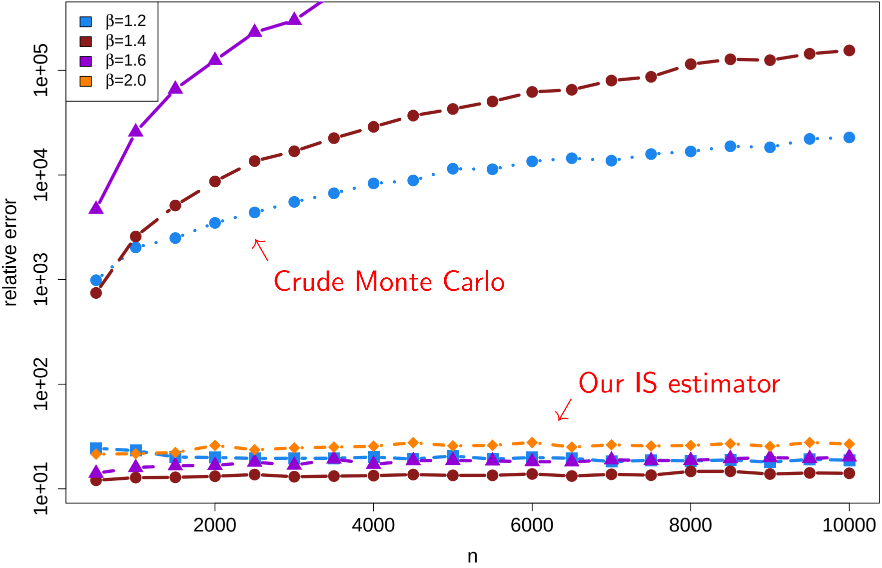

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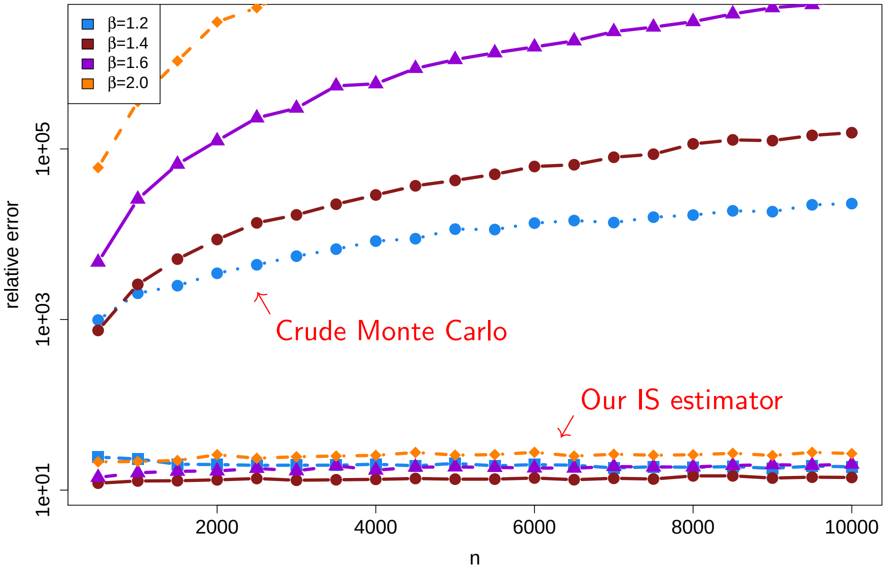

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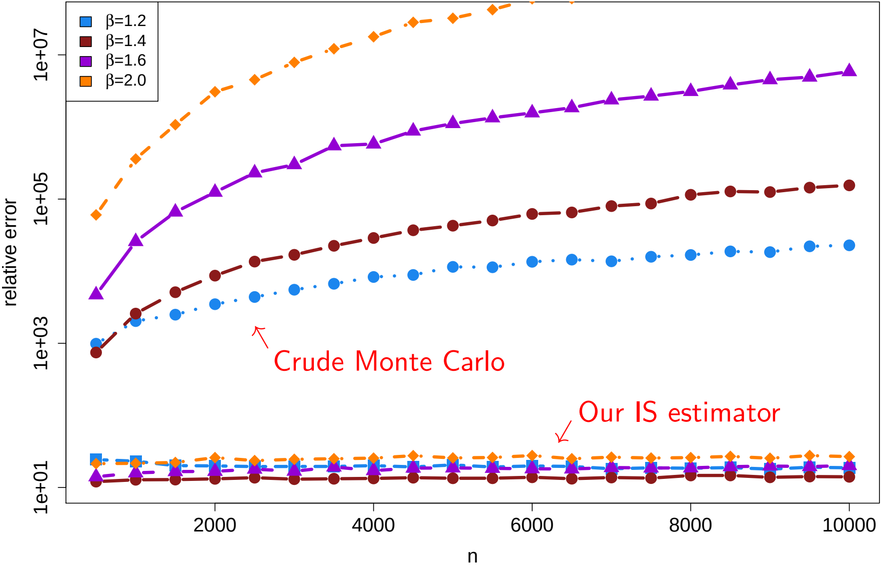

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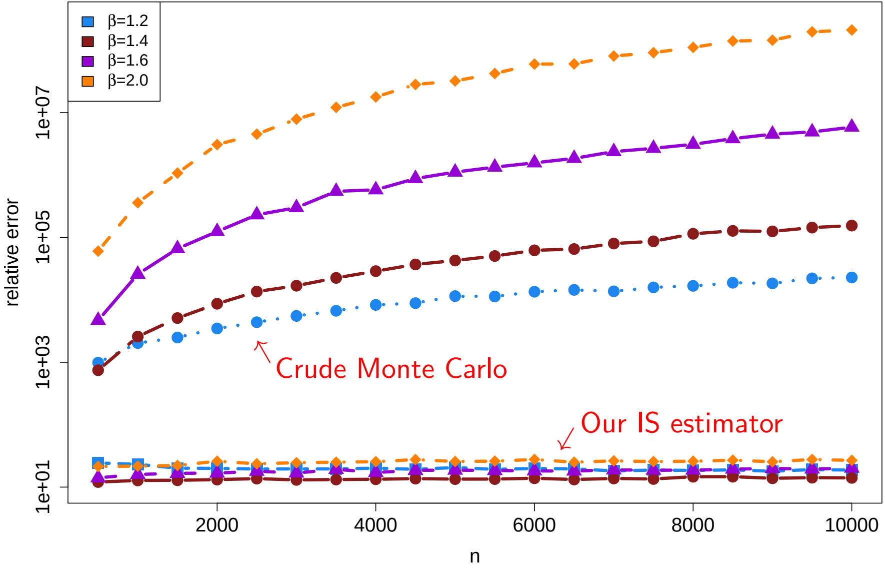

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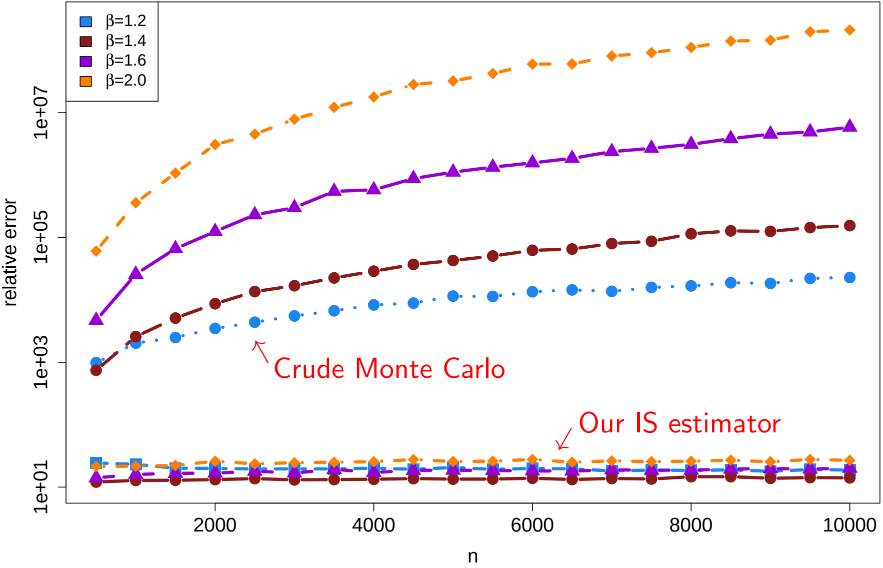
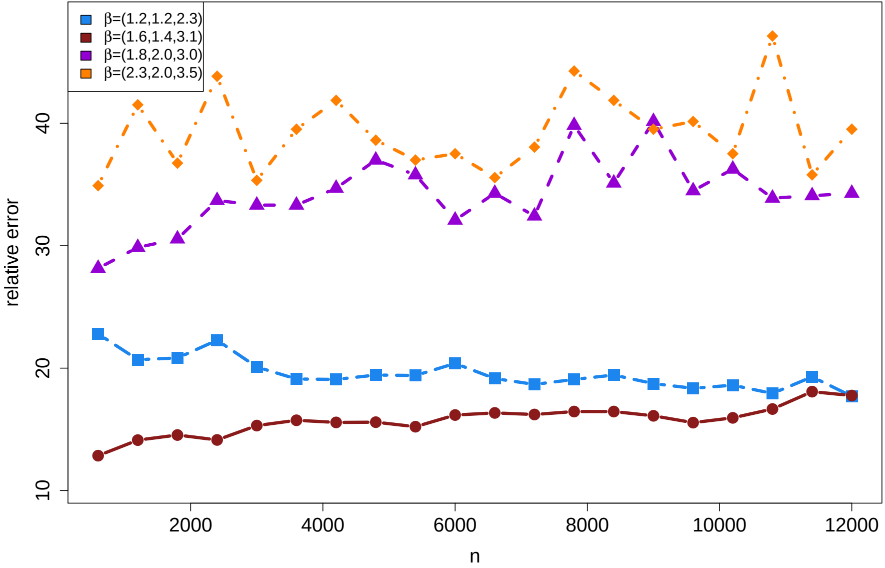


Numerical Results for Fluid Network Example



Part 3. Large Deviations for Weibull Tails

$$\mathbf{P}(X_i \geq x) = \exp(-x^{\alpha}),\ \ \alpha \in (0,1)$$

What's already known: LDP w.r.t. L_1 topology

Nina Gantert (1998)

- $\limsup_{n\to\infty} \frac{1}{L(n)n^{\alpha}} \log \mathbf{P}\left(\bar{S}_n \in A\right) \leq -\inf_{\xi\in A^-} I(\xi)$, $\qquad$ A^-: closure of A
- $\liminf_{n\to\infty} \frac{1}{L(n)n^{\alpha}} \log \mathbf{P}\left(\bar{S}_n \in A\right) \geq -\inf_{\xi\in A^\circ} I(\xi)$, $\qquad$ A°: interior of A

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- $\liminf_{n\to\infty} \frac{1}{L(n)n^\alpha} \log \mathbf{P}\left(\bar{S}_n \in A\right) \geq -\inf_{\xi\in A^\circ} I(\xi)$, A°: interior of A
- $I(\xi) = \begin{cases} \sum_{\{t:\xi(t)\neq\xi(t-)\}} \left(\xi(t)-\xi(t-)\right)^\alpha & \text{if } \xi \text{ is a nondecreasing step function} \\ \infty & \text{otherwise} \end{cases}$

What's already known: LDP w.r.t. L_1 topology

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- $\mathbf{P}\left(\bar{S}_n \in A\right) \lesssim \exp\left(-L(n)n^{\alpha}I(A^{-})\right)$, $\quad I(A^{-}) = \inf_{\xi \in A^{-}} I(\xi)$
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- I is a good rate function w.r.t. L_1 topology

  i.e., $d(\xi,\zeta) = \int_0^1 |\xi(s) - \zeta(s)| ds$

What's already known: LDP w.r.t. L_1 topology

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$\Rightarrow$ **Classical Tools in LD Theory Suffices**

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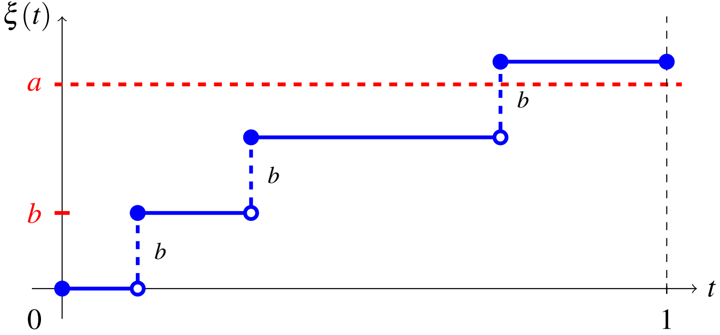
$\Rightarrow$ **Classical Tools in LD Theory Suffices**

BUT

L_1 topology is weak



- $A = \{\xi \in \mathbb{D} : \sup_{t\in[0,1]} \xi(t) \geq a, \sup_{t\in[0,1]} |\xi(t) - \xi(t-)| \leq b\}$
- $\mathbf{P}\left(\bar{S}_n \in A\right) \lesssim \exp\left(-L(n)n^{\alpha} I(A^-)\right)$, $\qquad I(A^-) = \inf_{\xi\in A^-} I(\xi)$
- $I(A^-) \stackrel{?}{=} b^{\alpha} + b^{\alpha} + b^{\alpha}$

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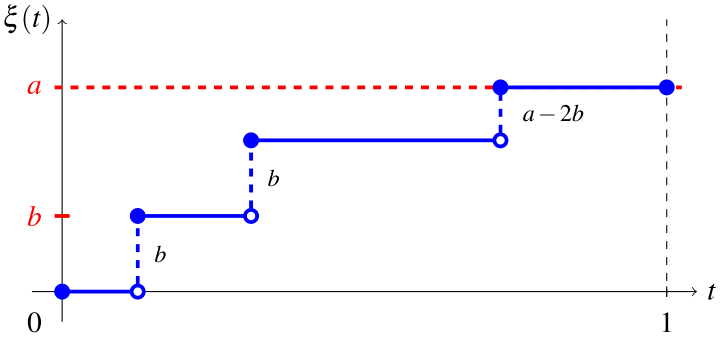

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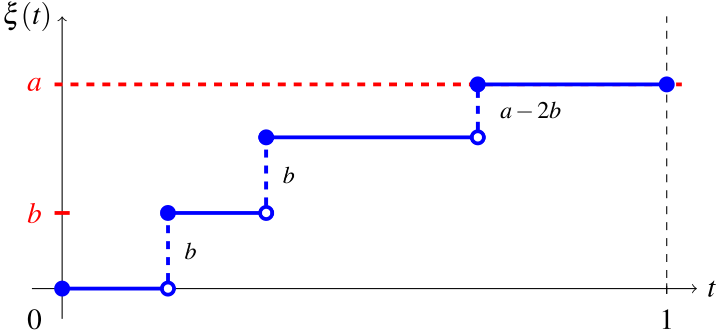

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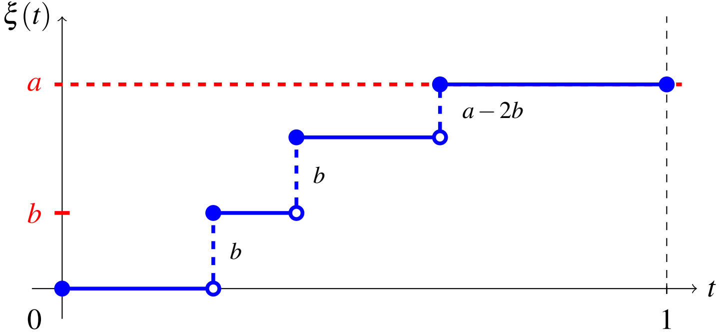

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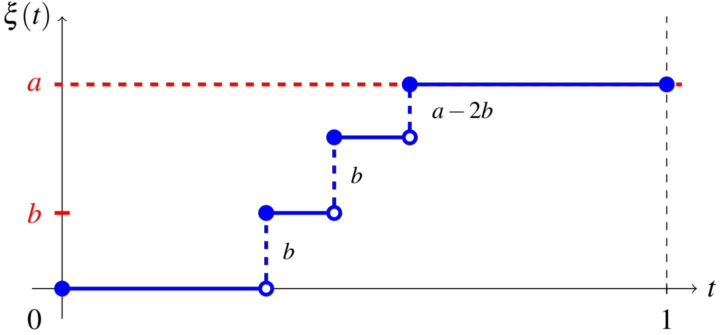

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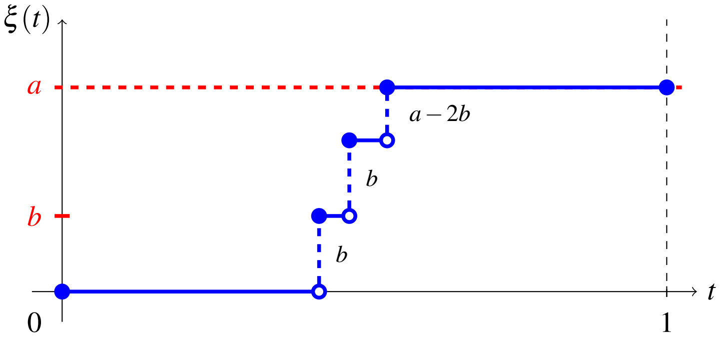

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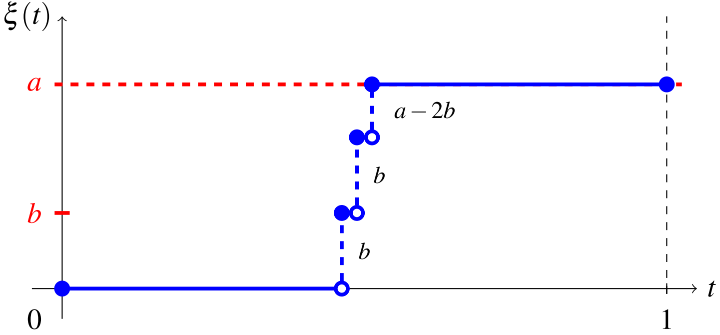

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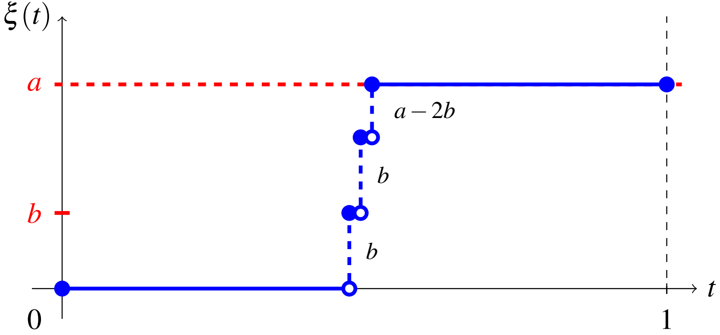

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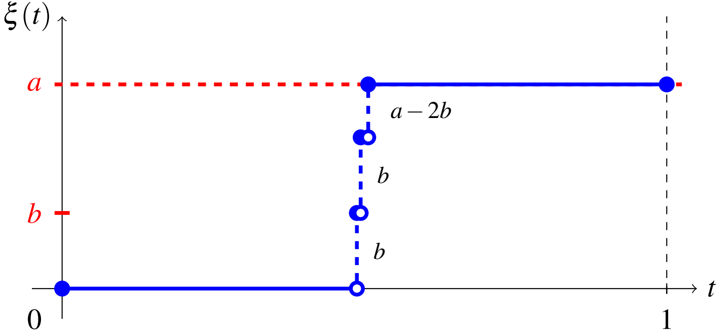

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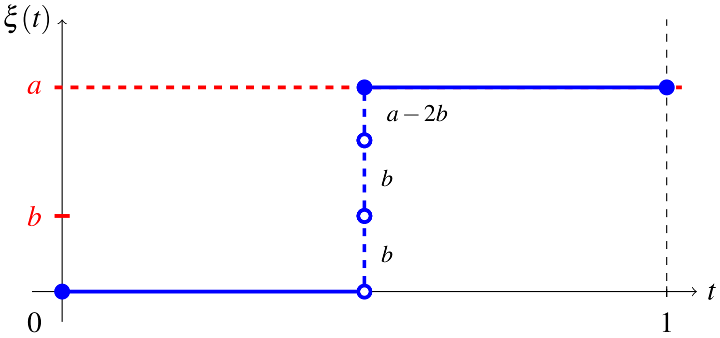

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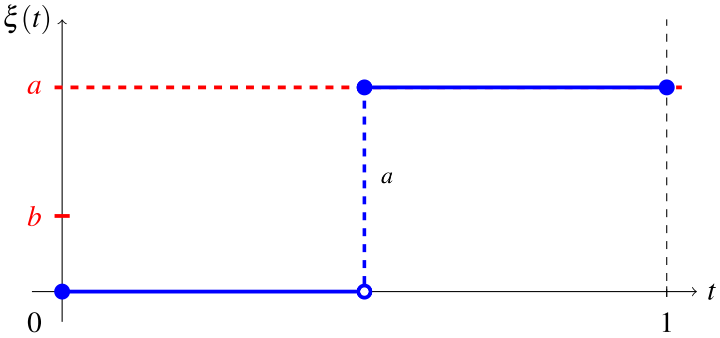

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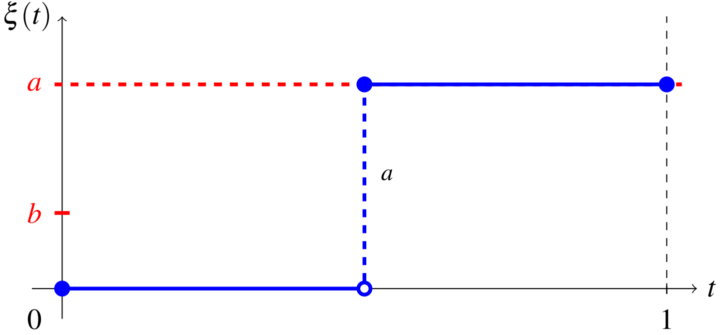

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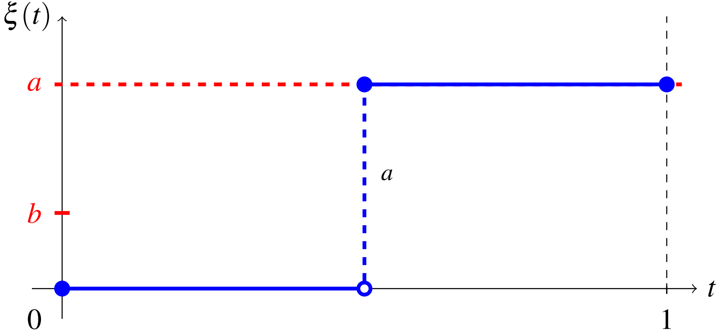

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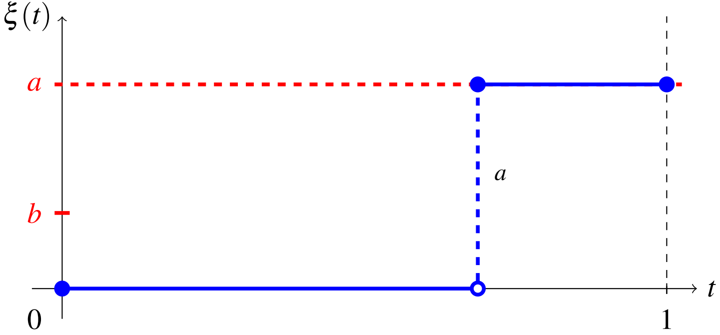

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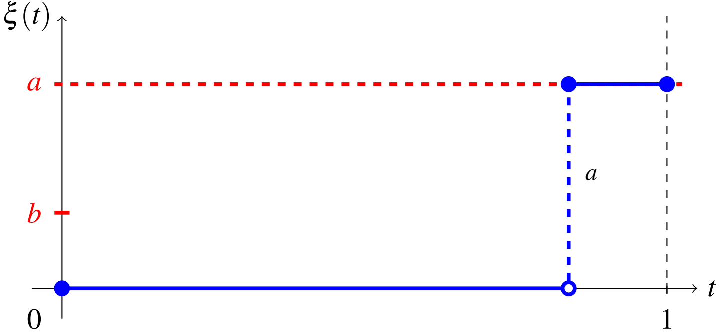

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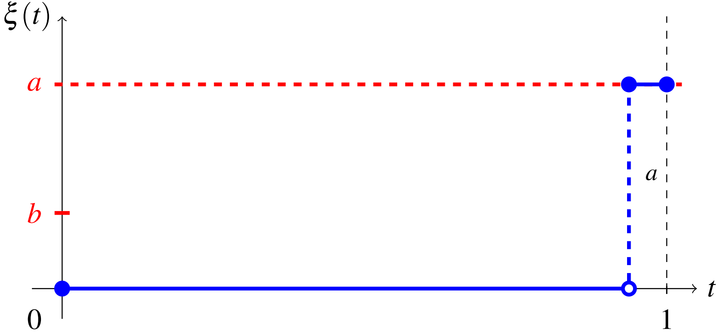

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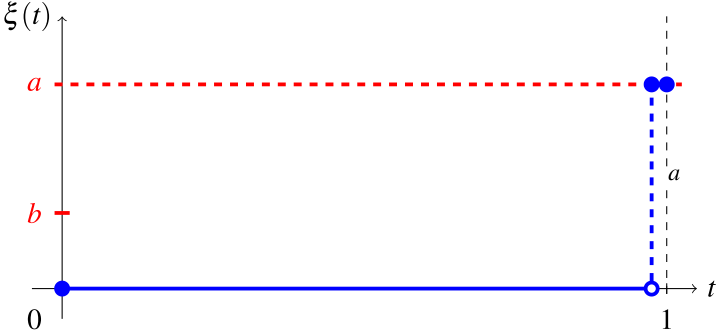

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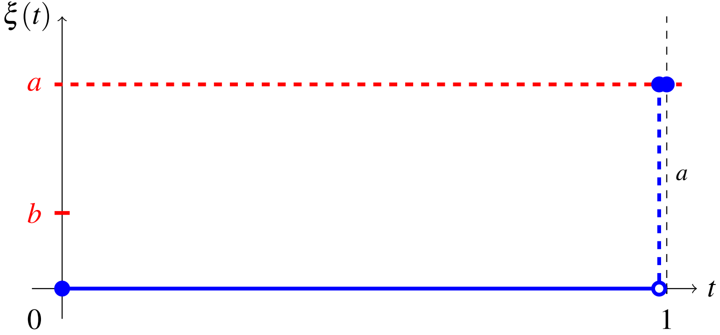

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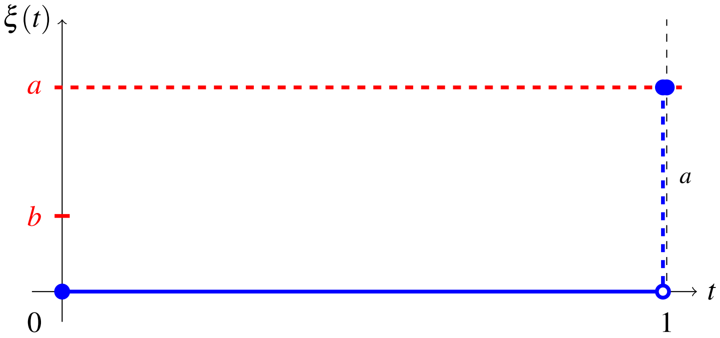

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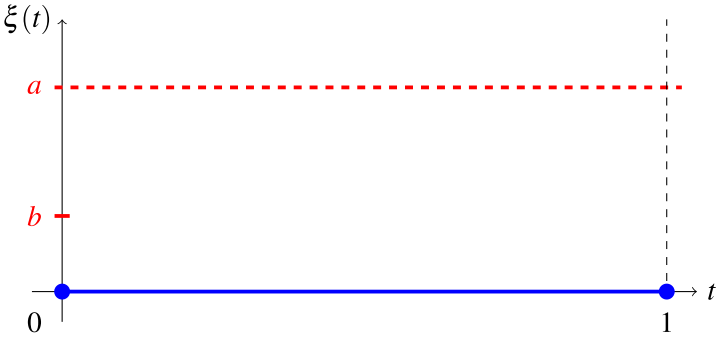

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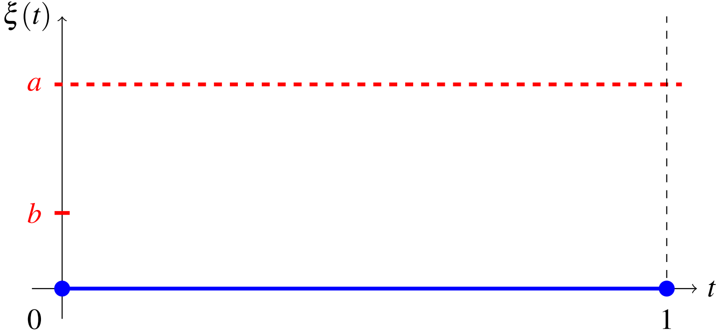

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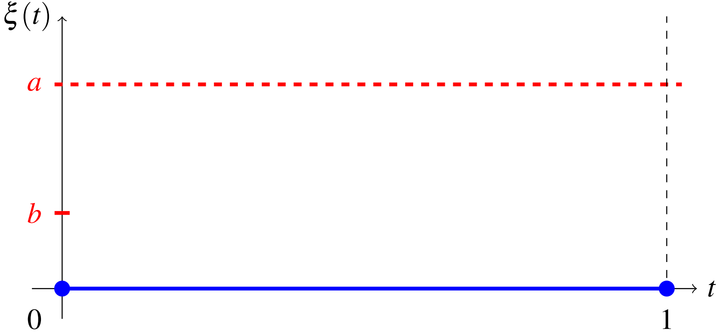

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No information!

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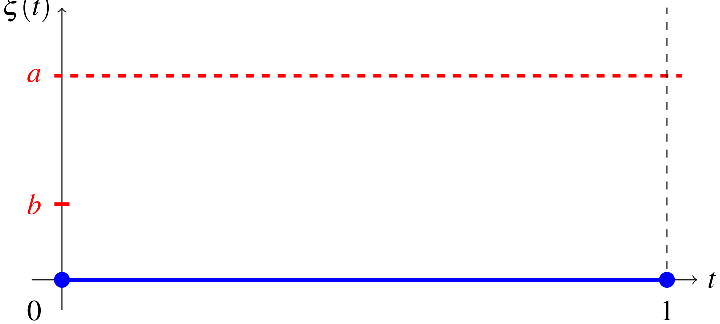

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Want a stronger topology, ideally J_1 topology!

LDP w.r.t. J_1 Topology is Impossible

Counterexample (Bazhba, Blanchet, R., Zwart 2017)

There exists a closed set $A \subseteq \mathbb{D}$ such that

$$\limsup_{n\to\infty} \frac{1}{\sqrt{n}} \log \mathbf{P}\left(\bar{S}_n \in A\right) \nleq -\inf_{\xi \in A} I(\xi)$$

- $L(n) = 1$ and $\alpha = \frac{1}{2}$ so that $\mathbf{P}(X_1 > x) = e^{-\sqrt{x}}$
- Paths in A have m increases of size $O\left(\frac{1}{m^2}\right)$, for some m

"Extended" Large Deviation Principle

Theorem (Bazhba, Blanchet, R., Zwart 2017)

$\bar{X}_n$ *satisfies an "extended LDP" w.r.t. J_1 topology, i.e.,*

$$\limsup_{n\to\infty} \frac{1}{L(n)n^{\alpha}} \log \mathbf{P}\left(\bar{X}_n(t) \in A\right) \le -\lim_{\varepsilon\to 0} \inf_{\xi\in A^{\varepsilon}} I(\xi)$$

$$\liminf_{n\to\infty} \frac{1}{L(n)n^{\alpha}} \log \mathbf{P}\left(\bar{X}_n(t) \in A\right) \ge -\inf_{\xi\in A^{\circ}} I(\xi)$$

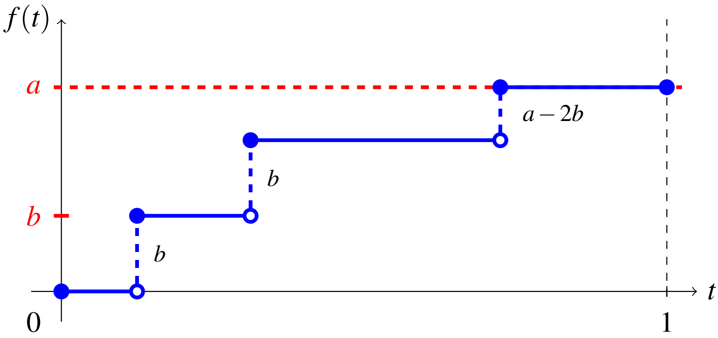
where

$$I(\xi) = \begin{cases} \sum_{\{t:\xi(t)\neq\xi(t^-)\}} \left(\xi(t) - \xi(t^-)\right)^{\alpha}, & \text{if } \xi \text{ is a nondecreasing pure jump function} \\ \infty, & o.w. \end{cases}$$

Corollary

If ϕ is Lipschitz, $\phi\left(\bar{X}_n\right)$ satisfies a LDP, if the resulting rate function is good.

Back to our old example



- $A = \{f \in \mathbb{D} : f$ crosses level a on $[0,1]$ & jump sizes $\leq b\}$
- $\mathbf{P}\left(\bar{S}_n \in A\right) \;\sim\; \exp\left(-n^{\alpha} I^*\right)$

  where $I^* = b^{\alpha} + b^{\alpha} + (a-2b)^{\alpha}$.

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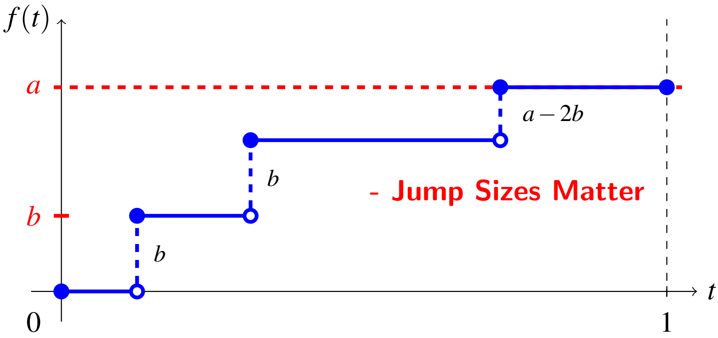

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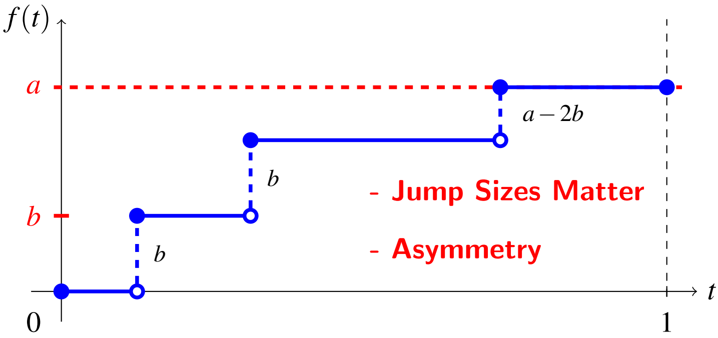

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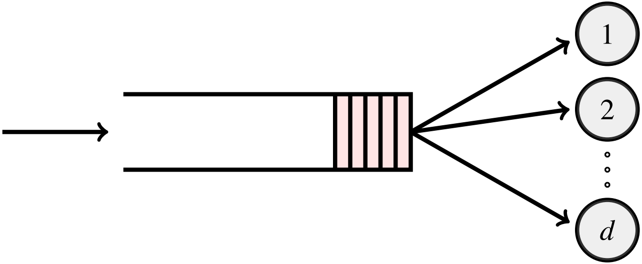
In case service time distribution is Weibull, i.e., $\mathbf{P}(S > x) \sim \exp(-x^{\alpha})$,



Tail Asymptotics? Most likely scenario?

So far, NOT EVEN a reasonable conjecture!

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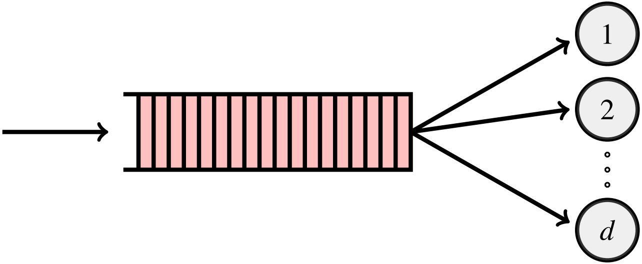

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Our Sample Path LDP Answers the Question!

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$$\mathbf{P}(Q(\gamma n) > n) \approx \mathbf{P}\left(\sup_{0 \le s \le \gamma} \left(\bar{M}_n(s) - \sum_{k=1}^{d} \bar{N}_n^{(k)}(s) \right) > 1 \right)$$

Our Sample Path LDP Answers the Question!

- $Q(t)$: length of GI/GI/d queue at time t

iid inter-arrival times

iid service times for server k

- $A(t) = A_1 + A_2 + \cdots + A_{\lfloor t \rfloor}$, $\qquad S^{(k)}(t) = S_1^{(k)} + S_2^{(k)} + \cdots + S_{\lfloor t \rfloor}^{(k)}$

inverse of $A(\cdot)$

inverse of $S^{(k)}(\cdot)$

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If service time distribution is Weibull, i.e., $\mathbf{P}(S_1 > x) \sim \exp(-x^{\alpha})$, then

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Solution to Open Question Posed by Whitt (2000) and Foss (2009)

More Specifically,

If $\gamma > \frac{1}{\lambda - \lfloor \lambda \rfloor}$: job/jump sizes (of the most likely scenario) are **symmetric**

- If $\lfloor \lambda \rfloor \leq \frac{\lambda - \alpha d}{1 - \alpha}$: smallest possible number of big jobs to block enough servers $\Rightarrow$ **same as the power law case**
- If $\lfloor \lambda \rfloor > \frac{\lambda - \alpha d}{1 - \alpha}$: larger number of moderately big jobs might be more likely $\Rightarrow$ **qualitatively different from the power law case**

If $\gamma < \frac{1}{\lambda - \lfloor \lambda \rfloor}$: job/jump sizes may be **asymmetric** (upto 3 different sizes)

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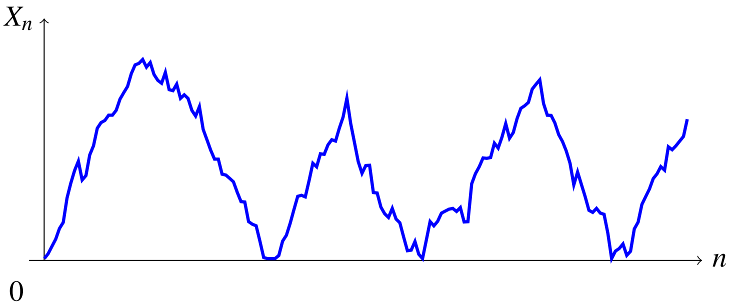
- **Goal:** sample path LD for $\bar{Y}_n$

Idea: View Regeneration Cycle as Heavy-Tailed Increment

- Recall: $\bar{Y}_n(t) = \frac{1}{n}\sum_{i=1}^{\lfloor nt \rfloor} f(X_i), \quad X_{n+1} = \max\{X_n + U_n, 0\}, \quad f(x) = x^p$



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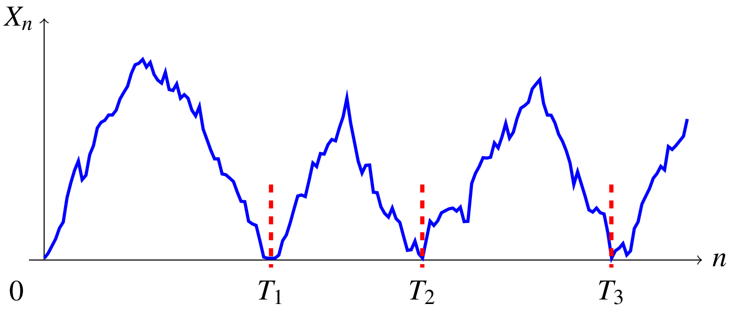
- $\{r_n\}_{n\geq 1}$: regeneration sequence, i.e., $\{X_{r_n}, \ldots, X_{r_{n+1}-1}\}_{n\geq 1}$ is i.i.d.



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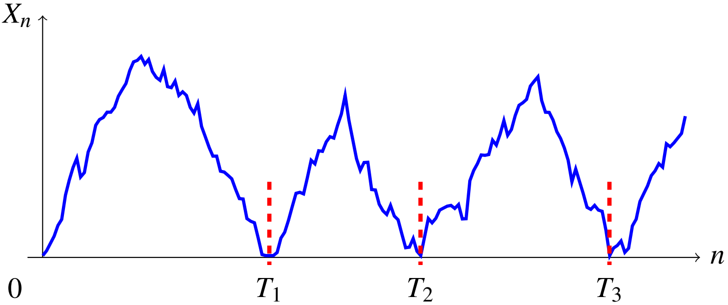

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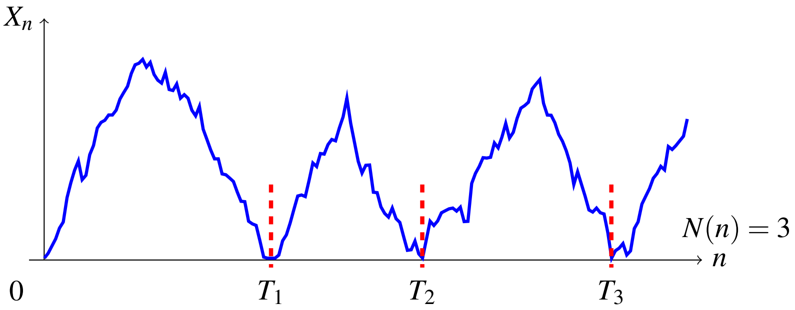

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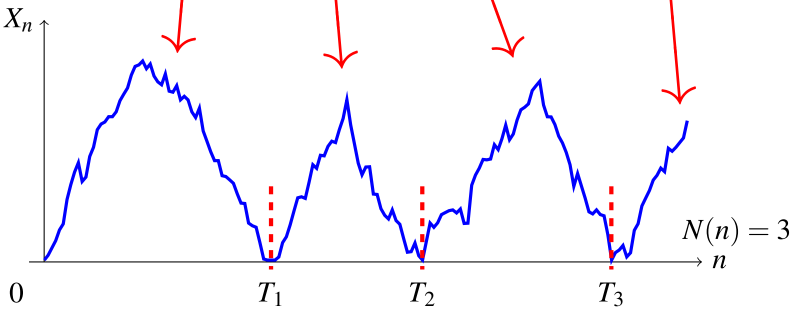

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Therefore,

"LDP for $\bar{Z}_n$ $\quad + \quad$ LDP for $\bar{V}_n$ $\quad \Longrightarrow \quad$ LDP for $\bar{Y}_n$"

LDP for $\bar{Z}_n$

- Recall $\bar{Z}_n(t) = \frac{1}{n}\left(W_1 + \cdots + W_{N(nt)}\right)$
- It turns out that $\mathbf{P}(W_1 \geq n) \approx \exp\left(-\mathcal{V}_0^* n^{1/(p+1)}\right)$ where

$$\mathcal{V}_0^* \triangleq \inf_{\xi \in A} \int_0^{\tau_0(\xi)\wedge T} \Lambda^*(\dot{\xi}(s))ds$$

$$A \triangleq \left\{\xi \in \mathbb{D} : \ \xi \in \mathcal{AC}, \ \xi(0) = 0, \ \int_0^{\tau_0(\xi)\wedge T} \{\Phi(\xi)(s)\}^p ds \geq 1\right\}$$

$$\tau_0(\xi) \triangleq \inf\{s \geq 0 : \xi(s) \leq 0\}$$

Skorkhod map

Therefore, LDP for $\bar{Z}_n$ can be obtained from our Weibull sample path LDP in Bazhba, Blanchet, Rhee, Zwart (2017).

LDP for $\bar{V}_n$

- Recall $\beta > 0$ is s.t. $\mathbf{E}\, e^{\beta U} = 1,\ \ \mathbf{E}\, U e^{\beta U} < \infty$

- $\lim_{n\to\infty} \frac{1}{n^{1/(p+1)}} \log \mathbf{P}_0(\bar{V}_n \geq x) = \lim_{n\to\infty} \frac{1}{n^{1/(p+1)}} \log \mathbf{P}_\pi(W_1 \geq x)$

- From this, we can prove that

$$\mathbf{P}(\bar{V}_n(1) \geq x) \approx \exp\left(-\mathcal{V}_\pi^* n^{1/(p+1)}\right)$$

where

$$\mathcal{V}_\pi^* \triangleq \inf_{\{y\in[0,\infty],\xi\in A\}} \left\{\beta y + \int_0^{\tau_0(\xi)\wedge T} \Lambda^*(\dot{\xi}(s))ds\right\}$$

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Summary

- **Systematic tools** for rare-event analysis of heavy-tailed systems

  - **Characterization of Catastrophe Principle**

  - **Strongly Efficient Rare-event Simulation Algorithm**