On Fairness of Systemic Risk Measures

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Outline

- Brief introduction to convex risk measures and general capital requirements

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- Brief introduction to Systemic Risk Measures
  - Aggregation functions
  - First aggregate, then inject capital
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  - **Not only inject cash but also random (***scenario-dependent***) capital injection** (*BFFM2018 Mathematical Finance*)

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- Brief introduction to Systemic Risk Measures
  - Aggregation functions
  - First aggregate, then inject capital
  - First inject capital, then aggregate
  - **Not only inject cash but also random (***scenario-dependent***) capital injection** (*BFFM2018 Mathematical Finance*)
- On Fairness of Systemic Risk Measures (*BFFM2019 this paper*)
  - Some basic conceptual questions on fairness and their solutions
  - Technical results

Risk Measures

Artzner, Delbaen, Eber and Heath (1999); Föllmer and Schied (2002); Frittelli and Rosazza Gianin (2002)

A monetary risk measure is a map

$$\eta : \mathcal{L}^0(\mathbb{R}) \to \mathbb{R}$$

that represents the **minimal (extra) capital needed to secure a financial position** with payoff $X \in \mathcal{L}^0(\mathbb{R})$, i.e. the minimal amount $m \in \mathbb{R}$ that must be added to X in order to make the resulting payoff at time T acceptable:

$$\eta(X) := \inf\{m \in \mathbb{R} \mid X + m \in \mathbb{A}\},$$

where the acceptance set $\mathbb{A} \subseteq \mathcal{L}^0(\mathbb{R})$ is assumed to be monotone.

Example of acceptance set:

$$\mathbb{A} := \{Z \in \mathcal{L}^0(\mathbb{R}) \mid \mathbb{E}[Z] \geq B\}, B \in \mathbb{R}.$$

Coherent / Convex Risk Measures/ Cash Additivity

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The characterizing feature of these monetary maps is **cash additivity**:

$$\eta(X + m) = \eta(X) - m, \quad \text{for all } m \in \mathbb{R}.$$

Under the assumption that the set $\mathbb{A}$ is convex (resp. is a convex cone) the maps η are convex (resp. convex and positively homogeneous) and are called **convex (resp. coherent) risk measures**.

General Capital Requirement

Frittelli and Scandolo 2006

$$\eta(X) \triangleq \inf\{m \in \mathbb{R} \mid X + m1 \in \mathbb{A}\}, \quad \mathbb{A} \subseteq \mathcal{L}^0(\mathbb{R})$$

Why should we consider **only** “money” as safe capital ?

One should be more liberal and **permit the use of other financial assets** (other than the bond $:= 1$), in an appropriate set $\mathcal{C}$ of *safe* instruments, **to hedge the position** X.

Definition

The general capital requirement is

$$\eta(X) \triangleq \inf\{\pi(Y) \in \mathbb{R} \mid Y \in \mathcal{C},\, X + Y \in \mathbb{A}\},$$

for some evaluation functional $\pi : \mathcal{C} \to \mathbb{R}$.

Systemic Risk Measure

Consider a system of N interacting financial institutions and a vector $\mathbf{X} = (X^1, \ldots, X^N) \in \mathcal{L}^0(\mathbb{R}^N) := \mathcal{L}^0(\Omega, \mathcal{F}; \mathbb{R}^N)$ of associated risk factors (future values of positions) at a given future time horizon T.

- In this paper we are interested in **real-valued** systemic risk measures:

$$\rho : \mathcal{L}^0(\mathbb{R}^{\mathbb{N}}) \to \overline{\mathbb{R}}$$

  that evaluates the risk $\rho(\mathbf{X})$ of the complete financial system $\mathbf{X}$.

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- Initially, many of the SRM in the literature were of the form

$$\rho(\mathbf{X}) = \eta(\Lambda(\mathbf{X})),$$

  where $\eta : \mathcal{L}^0(\mathbb{R}) \to \overline{\mathbb{R}}$ is a univariate risk measure and

$$\Lambda : \mathbb{R}^N \to \mathbb{R}$$

  is an aggregation rule that aggregates the N-dimensional risk factor $\mathbf{X}$ into a univariate risk factor $\Lambda(\mathbf{X})$ representing the total risk in the system.

Examples of aggregation rule

- $\Lambda(\mathbf{x}) = \sum_{n=1}^{N} x_n, \quad \mathbf{x} = (x_1, ..., x_N) \in \mathbb{R}^N.$
- $$\Lambda(\mathbf{x}) = \sum_{n=1}^{N} -x_n^- \quad \text{or} \quad \Lambda(\mathbf{x}) = \sum_{n=1}^{N} -(x_n - d_n)^-,\ d_n \in \mathbb{R}$$
- $$\Lambda(\mathbf{x}) = \sum_{n=1}^{N} -\exp\left(-\alpha_n x_n^-\right), \quad \alpha_n \in \mathbb{R}_+$$
  $$\Lambda(\mathbf{x}) = \sum_{n=1}^{N} -\exp\left(-\alpha_n x_n\right), \quad \alpha_n \in \mathbb{R}_+$$
- $$\Lambda(\mathbf{x}) = \sum_{n=1}^{N} u_n(x_n)$$

  where $u_n : \mathbb{R} \to \mathbb{R}$ are utility functions.

First aggregate, then inject cash

$$\rho(\mathbf{X}) = \eta(\Lambda(\mathbf{X})),$$

If η is a convex (cash additive) risk measure then we can rewrite such ρ as

$$\rho(\mathbf{X}) \triangleq \inf\{m \in \mathbb{R} \mid \Lambda(\mathbf{X}) + m \in \mathbb{A}\}.$$

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- For an axiomatic approach for this type of SRM see Chen Iyengar Moallemi 2013 and Kromer Overbeck Zilch 2013 and the references therein. Acharya et al. 2010, Adrian Brunnermeier 2011, Cheridito Brunnermeier 2014, Gauthier Lehar Souissi 2010, Hoffmann Meyer-Brandis Svindland 2014, Huang Zhou Zhu 2009, Lehar 2005.

In BFFM18: First inject cash, then aggregate

$$\rho(\mathbf{X}) \triangleq \inf\{\sum_{n=1}^{N} m_n \in \mathbb{R} \mid \mathbf{m} = (m_1, ..., m_N) \in \mathbb{R}^N; \Lambda(\mathbf{X}+\mathbf{m}) \in \mathbb{A}\}.$$

- The amount m_n is added to the financial position X^n before the corresponding total loss $\Lambda(\mathbf{X}+\mathbf{m})$ is computed.
- $\rho(X)$ **is the minimal capital that secures the aggregated system by injecting the capital into the single institutions before aggregating the individual risks.**

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- Also applied to shortfall systemic risk measures by Armenti-Crepey-Drapeau-Papapantoleon (2018).
- ρ delivers at the same time a measure of total systemic risk and a potential **ranking** of the institutions in terms of systemic riskiness.

Second feature of BFFM18: Random allocation

We allow to add to $\mathbf{X}$ not only a vector $\mathbf{m} = (m_1, ..., m_N) \in \mathbb{R}^N$ of cash but a random vector

$$\mathbf{Y} \in \mathcal{C}_{\mathbb{R}} := \{\mathbf{Y} \in L^0(\mathbb{R}^N) \mid \sum_{n=1}^{N} Y^n \in \mathbb{R}\},$$

so that $\rho(\mathbf{X})$ is the minimal cash $\sum_{n=1}^{N} Y^n \in \mathbb{R}$ needed today to secure the system by distributing the capital at time T among $(X^1, ..., X^N)$:

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- In general the allocation $Y^i(\omega)$ to institution i does not need to be decided today but depends on the scenario ω realized at time T.
- For $\mathcal{C} = \mathbb{R}^N$ the situation corresponds to the previous case where the distribution is already determined today.
- For $\mathcal{C} = \mathcal{C}_{\mathbb{R}}$ the distribution can be chosen freely depending on the scenario ω realized in T (including negative amounts, i.e. **full cross-subsidization** or **risk-sharing**).

Grouping Example

For a partition of $\{1, ..., N\}$ in h groups we consider the set $\mathcal{C}^{(h)} \subseteq \mathcal{C}_{\mathbb{R}}$

$$\mathcal{C}^{(h)} = \left\{ \mathbf{Y} \in \mathcal{L}^0(\mathbb{R}^N) \mid \exists\ d = (d_1, \cdots, d_h) \in \mathbb{R}^h : \sum_{i \in I_m} Y^i = d_m, m = 1, .., h \right\}$$

- the values $(d_1, \cdots, d_h)$ may change, but the number of elements in each of the h groups I_m is fixed.
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- $h = N$, (exactly N groups), then $\mathcal{C}^{(h)} = \mathbb{R}^N$ corresponds to the deterministic case
- $h = 1$ (only one group) then $\mathcal{C}^{(h)} = \mathcal{C}_{\mathbb{R}}$. Completely arbitrary random injection $\mathbf{Y}$ with the only requirement $\sum_{n=1}^{N} Y^n \in \mathbb{R}$

Dependence can be taken into account

- Allowing **random** allocations $\mathbf{Y} \in \mathcal{C} \subseteq \mathcal{C}_{\mathbb{R}}$, that might differ from scenario to scenario, the systemic risk measure will take the dependence structure of the components of $\mathbf{X}$ into account even though acceptable positions might be defined in terms of the marginal distributions of X^n, $n = 1, ..., N$, only.
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- This fact allows to reduce the total systemic risk.
- For example: $\Lambda(\mathbf{x}) = \sum_{n=1}^{N} u_n(x^n)$, $u_n : \mathbb{R} \to \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^N$.

$$\mathbb{A} := \{Z \in \mathcal{L}^0(\mathbb{R}) \mid \mathbb{E}[Z] \geq B\}, B \in \mathbb{R}.$$

- $\mathbf{X} + \mathbf{Y} \in \mathcal{L}^0(\mathbb{R}^N)$ is acceptable if and only if $\Lambda(\mathbf{X} + \mathbf{Y}) \in \mathbb{A}$, i.e.

$$\mathbb{E}\left[\sum_{n=1}^{N} u_n(X^n + Y^n)\right] \geq B$$

- If $\mathcal{C} = \mathbb{R}^N$ (i.e. $\mathbf{Y} = \mathbf{m} \in \mathbb{R}^N$) then $\rho(\mathbf{X})$ depends on the marginal distributions of $\mathbf{X}$ only.

Fairness (BFFM19). Definitions and Assumptions

$$\rho(\mathbf{X}) := \inf_{\mathbf{Y} \in \mathcal{C}} \left\{ \sum_{n=1}^{N} Y^n \mid \mathbb{E}\left[\sum_{n=1}^{N} u_n(X^n + Y^n) \right] \geq B \right\},$$

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2. $u_n : \mathbb{R} \to \mathbb{R}$ is increasing, strictly concave, differentiable and satisfies the Inada conditions

$$u_n'(-\infty) \triangleq \lim_{x\to-\infty} u_n'(x) = +\infty,\ u_n'(+\infty) \triangleq \lim_{x\to+\infty} u_n'(x) = 0.$$

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3. $B < \sum_{n=1}^{N} u_n(+\infty)$

$\rho = \rho_B$ is well defined and has good properties

$$\rho(\mathbf{X}) := \inf_{\mathbf{Y} \in \mathcal{C}} \left\{ \sum_{n=1}^{N} Y^n \mid \mathbb{E}\left[\sum_{n=1}^{N} u_n(X^n + Y^n) \right] \geq B \right\}$$

Proposition

The map $\rho : M^{\Phi} \to \mathbb{R} \cup \{+\infty\}$ is finitely valued, monotone decreasing, convex, continuous and subdifferentiable on the Orlicz Heart $M^{\Phi} = dom(\rho)$.

$$\rho : M^{\Phi} \to \mathbb{R}$$

where we skip the details on the Orlicz setting

Optimal Allocation and Risk Allocation

$$\rho(\mathbf{X}) := \inf_{\mathbf{Y}\in\mathcal{C}} \left\{ \sum_{n=1}^{N} Y^n \mid \mathbb{E}\left[\sum_{n=1}^{N} u_n(X^n + Y^n)\right] \geq B \right\}$$

Definition

(i) The scenario dependent allocation $\mathbf{Y_X} = (Y^n_{\mathbf{X}})_n \in \mathcal{C}$ is an **optimal allocation** to $\rho(\mathbf{X})$, if

$$\mathbb{E}\left[\sum_{n=1}^{N} u_n(X^n + Y^n_{\mathbf{X}})\right] \geq B \quad \text{and} \quad \rho(\mathbf{X}) = \sum_{n=1}^{N} Y^n_{\mathbf{X}}.$$

(ii) A vector $(\rho^n(\mathbf{X}))_n \in \mathbb{R}^N$ is a **risk allocation** for $\rho(\mathbf{X})$ if

$$\rho(\mathbf{X}) = \sum_{n=1}^{N} \rho^n(\mathbf{X}).$$

Key qualitative questions on fairness

1. When is the *systemic valuation* $\rho(\mathbf{X})$ *and its random allocation* $\mathbf{Y_X} = (Y^n_{\mathbf{X}})_n \in \mathcal{C}_{\mathbb{R}}$ fair from the point of view of the *whole system*?

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2. When is a *risk allocation* $(\rho^n(\mathbf{X}))_n \in \mathbb{R}^N$ of $\rho(\mathbf{X})$ fair from the point of view of the *whole system*?

3. When are the *systemic allocation* $\mathbf{Y_X} = (Y^n_{\mathbf{X}})_n \in \mathcal{C}_{\mathbb{R}}$ and the *risk allocation* $(\rho^n(\mathbf{X}))_n \in \mathbb{R}^N$ associated to $\rho(\mathbf{X})$, fair from the point of view of *each individual bank*?

Preview of the answer

Let $\mathbf{Y_X} = (Y^1_{\mathbf{X}}, \cdots, Y^N_{\mathbf{X}})$ be an optimal random allocation for $\rho(\mathbf{X})$

$$\rho(\mathbf{X}) = \sum_{n=1}^{N} Y^n_{\mathbf{X}}.$$

Let $\mathbf{Q_X} = (Q^1_{\mathbf{X}}, \cdots, Q^N_{\mathbf{X}})$ be the optimal solution of the **dual problem** for $\rho(\mathbf{X})$, then

$$\rho^n(\mathbf{X}) = \mathbb{E}_{Q^n_{\mathbf{X}}}\left[Y^n_{\mathbf{X}}\right], \quad n = 1, \cdots, N,$$

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Technical questions:

a) What is the dual representation of $\rho(\mathbf{X})$?

b) When an optimal solution $\mathbf{Q_X}$ to the dual problem exists?

c) When an optimal random allocation $\mathbf{Y_X}$ exists?

Dual representation

Theorem

For any $\mathbf{X} \in M^{\Phi}$,

$$\rho(\mathbf{X}) = \max_{\mathbf{Q} \in \mathcal{D}} \left\{ \sum_{n=1}^{N} \mathbb{E}_{Q^n}[-X^n] - \alpha_B(\mathbf{Q}) \right\}, \tag{1}$$

$$\alpha_B(\mathbf{Q}) \quad = \quad \sup_{\mathbf{Z} \in M^{\Phi}} \left\{ \sum_{n=1}^{N} \mathbb{E}_{\mathbf{Q}^n}[-Z^n] \mid \sum_{n=1}^{N} \mathbb{E}[u_n(Z^n)] \geq B \right\},$$

$$\mathcal{D} := dom(\alpha_B) \cap \left\{ \frac{d\mathbf{Q}}{d\mathbb{P}} \in L_{+}^{\Phi^*} \mid Q^n(\Omega) = 1, \sum_{n=1}^{N} (\mathbb{E}_{Q^n}[Y^n] - Y^n) \leq 0 \ \forall \mathbf{Y} \in \mathcal{C} \right\}$$

The maximizer $\mathbf{Q_X} = (Q^1_{\mathbf{X}}, \cdots, Q^N_{\mathbf{X}})$ in (1) exists and is unique.

Additionally, an optimal allocation $\mathbf{Y_X}$ *exists.*

Penalty function

For $\alpha_B(\mathbf{Q}) < +\infty$, we have

$$\alpha_B(\mathbf{Q}) = \sum_{n=1}^{N} \mathbb{E}\left[\frac{dQ^n}{d\mathbb{P}} v_n'\left(\widehat{\lambda}\frac{dQ^n}{d\mathbb{P}}\right)\right],$$

where where $\widehat{\lambda} > 0$ is the unique solution of the equation

$$-B + \sum_{n=1}^{N} \mathbb{E}\left[v_n\left(\lambda\frac{dQ^n}{d\mathbb{P}}\right)\right] - \lambda \sum_{n=1}^{N} \mathbb{E}\left[\frac{dQ^n}{d\mathbb{P}} v_n'\left(\lambda\frac{dQ^n}{d\mathbb{P}}\right)\right] = 0.$$

Here, v_n is the convex conjugate function $v_n(y) := \sup_{x\in\mathbb{R}}\{u_n(x) - xy\}$. Note that $\widehat{\lambda}$ will depend on B, $(u_n)_{n=1,\cdots,N}$ and $\left(\frac{dQ_n}{d\mathbb{P}}\right)_{n=1,\cdots,N}$.

Back to the fairness questions

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  - Optimal random allocations: $\mathbf{Y_X} = (Y_{\mathbf{X}}^1, \cdots, Y_{\mathbf{X}}^N)$
  - Associated risk allocations: $\rho^n(\mathbf{X}) = \mathbb{E}_{Q_{\mathbf{X}}^n}\left[Y_{\mathbf{X}}^n\right]$ such that $\sum_{n=1}^N \rho^n(\mathbf{X}) = \rho(\mathbf{X})$

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- Recall:

$$\rho(\mathbf{X}) = \rho_B(\mathbf{X}) = \inf_{\mathbf{Y}\in\mathcal{C}} \left\{ \sum_{n=1}^N Y^n \mid \mathbb{E}\left[\sum_{n=1}^N u_n(X^n + Y^n)\right] \geq B \right\}$$

- First, we show how the systemic probabilities $Q^n_{\mathbf{X}}$ appear naturally.

Cash additivity

Let $\mathbf{Z} \in \mathcal{C}_{\mathbb{R}}$ such that $\mathbf{Y} \in \mathcal{C} \iff \mathbf{Y} - \mathbf{Z} \in \mathcal{C}$, then it's easy to show that ρ satisfies the **cash additivity property**:

$$\rho(\mathbf{X} + \mathbf{Z}) = \rho(\mathbf{X}) - \sum_{n=1}^{N} Z^n.$$

In particular, applied to $\mathbf{Z} = \varepsilon \mathbf{V}$, gives the **marginal risk contribution property**:

$$\frac{d}{d\varepsilon} \rho(\mathbf{X} + \varepsilon \mathbf{V})|_{\varepsilon=0} = -\sum_{n=1}^{N} V^n.$$

Note that it applies to $\mathbf{V} \in \mathbb{R}^N$.

Marginal Risk Contribution

The marginal risk contribution $\frac{d}{d\varepsilon}\rho(\mathbf{X}+\varepsilon\mathbf{V})|_{\varepsilon=0}$ is an important quantity which describes the sensitivity of the risk of $\mathbf{X}$ with respect to the impact $\mathbf{V} \in L^0(\mathbb{R}^N)$. The previous property cannot be immediately generalized to the case of random vectors $\mathbf{V}$ as $\sum_{n=1}^{N} V^n \notin \mathbb{R}$ in general.

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In the following, we obtain the general local version of cash additivity, which allows for a random setting.

Proposition

Let $\mathbf{X}$ and $\mathbf{V} \in M^\Phi$. Let $\mathbf{Q_X}$ be the optimal solution to the dual problem associated to $\rho(\mathbf{X})$ and assume that $\rho(\mathbf{X}+\varepsilon\mathbf{V})$ is differentiable with respect to ε at $\varepsilon = 0$, and $\frac{d\mathbf{Q}_{\mathbf{X}+\varepsilon\mathbf{V}}}{d\mathbb{P}} \to \frac{d\mathbf{Q_X}}{d\mathbb{P}}$ in $\sigma^*(L^{\Phi^*}, M^\Phi)$, as $\varepsilon \to 0$. Then,

$$\frac{d}{d\varepsilon}\rho(\mathbf{X}+\varepsilon\mathbf{V})|_{\varepsilon=0} = -\sum_{n=1}^N \mathbb{E}_{Q_{\mathbf{X}}^n}[V^n].$$

Marginal Risk Contribution, proof

Proof.

As the penalty function α_B does not depend on $\mathbf{X}$, we deduce

$$
\begin{aligned}
\frac{d}{d\varepsilon}\rho(\mathbf{X}+\varepsilon\mathbf{V})|_{\varepsilon=0} &= \frac{d}{d\varepsilon}\left\{\sum_{n=1}^{N}\mathbb{E}_{Q^n_{\mathbf{X}+\varepsilon\mathbf{V}}}[-X^n-\varepsilon V^n] - \alpha_B(\mathbf{Q}_{\mathbf{X}+\varepsilon\mathbf{V}})\right\}|_{\varepsilon=0} \\
&= \frac{d}{d\varepsilon}\left\{\sum_{n=1}^{N}\mathbb{E}_{Q^n_{\mathbf{X}+\varepsilon\mathbf{V}}}[-X^n] - \alpha_B(\mathbf{Q}_{\mathbf{X}+\varepsilon\mathbf{V}})\right\}|_{\varepsilon=0} \\
&\quad + \sum_{n=1}^{N}\frac{d}{d\varepsilon}\left(\varepsilon\mathbb{E}_{Q^n_{\mathbf{X}+\varepsilon\mathbf{V}}}[-V^n]\right)|_{\varepsilon=0} \\
&= 0 + \sum_{n=1}^{N}\lim_{\varepsilon\to 0}\mathbb{E}_{Q^n_{\mathbf{X}+\varepsilon\mathbf{V}}}[-V^n] = \sum_{n=1}^{N}\mathbb{E}_{Q^n_{\mathbf{X}}}[-V^n],
\end{aligned}
$$

by using the optimality of $\mathbf{Q}_{\mathbf{X}}$ and the differentiability of $\rho(\mathbf{X}+\varepsilon\mathbf{V})$, while the last equality is guaranteed by the convergence of $\frac{d\mathbf{Q}_{\mathbf{X}+\varepsilon\mathbf{V}}}{d\mathbb{P}}$. □

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In practice, the mechanism can be described as a **default fund** as in the case of a CCP.

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Then, at time T, this exact same amount would be redistributed among the banks according to the optimal scenario-dependent allocations $Y_{\mathbf{X}}^n$'s satisfying $\sum_{n=1}^{N} Y_{\mathbf{X}}^n = \rho(\mathbf{X})$, so that the fund acts as a **clearing house**.

Implementation of the scenario-dependent allocation (b)

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At time 0, a capital requirement $\rho^n(\mathbf{X})$ is imposed on each bank n, $n = 1, ..., N$.

Then, at time T, a risk sharing mechanism takes place: each bank provides (if negative) or collects (if positive) the amount $Y^n_{\mathbf{X}} - \rho^n(\mathbf{X})$. Note that the financial position of bank n at time T is $X^n + \rho^n(\mathbf{X}) + (Y^n_{\mathbf{X}} - \rho^n(\mathbf{X})) = X^n + Y^n_{\mathbf{X}}$ as required.

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Further, this risk sharing mechanism is made possible because of the **clearing property** $\sum_{n=1}^N (Y^n_{\mathbf{X}} - \rho^n(\mathbf{X})) = 0$ which follows from $\sum_{n=1}^N Y^n_{\mathbf{X}} = \rho(\mathbf{X})$ and the full risk allocation requirement $\sum_{n=1}^N \rho^n(\mathbf{X}) = \rho(\mathbf{X})$.

Property of the systemic risk allocation

We can introduce other systemic risk measures in terms of vectors of probability measures **Q** by

$$\rho^{\mathbf{Q}}(\mathbf{X}) := \inf_{\mathbf{Y}\in\mathcal{L}} \left\{ \sum_{n=1}^{N} \mathbb{E}_{Q^n}\left[Y^n\right] \mid \mathbb{E}\left[\sum_{n=1}^{N} u_n(X^n + Y^n)\right] \geq B \right\},$$

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where we **do not assume** that $\mathbf{Y}$ is in $C_{\mathbb{R}}$.
Then, we have

$$\sum_{n=1}^{N} Y_{\mathbf{X}}^n = \rho(\mathbf{X}) = \rho^{\mathbf{Q}_\mathbf{X}}(\mathbf{X}) = \sum_{n=1}^{N} \mathbb{E}_{Q_{\mathbf{X}}^n}[Y_{\mathbf{X}}^n],$$

which gives a natural risk allocation.

Associated maximization problems

$$\pi_A(\mathbf{X}) := \sup_{\mathbf{Y}\in\mathcal{C}\subset\mathcal{C}_{\mathbb{R}}} \left\{ \mathbb{E}\left[\sum_{n=1}^{N} u_n(X^n + Y^n)\right] \mid \sum_{n=1}^{N} Y^n \leq A \right\}.$$

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Then, one can show that

$$B = \pi_A(\mathbf{X}) \text{ if and only if } A = \rho_B(\mathbf{X}),$$

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Introducing the maximization problems

$$\pi^{\mathbf{Q}}(\mathbf{X}) = \pi_A^{\mathbf{Q}}(\mathbf{X}) := \sup_{\mathbf{Y}\in\mathcal{L}} \left\{ \mathbb{E}\left[\sum_{n=1}^{N} u_n(X^n + Y^n)\right] \mid \sum_{n=1}^{N} \mathbb{E}_{Q^n}\left[Y^n\right] \leq A \right\},$$

then, the optimizer $\mathbf{Q}_{\mathbf{X}} = (Q^1_{\mathbf{X}}, \cdots Q^N_{\mathbf{X}})$ satisfies:

$$\rho_B(\mathbf{X}) = \rho_B^{\mathbf{Q}_{\mathbf{X}}}(\mathbf{X}), \;\; \pi_A(\mathbf{X}) = \pi_A^{\mathbf{Q}_{\mathbf{X}}}(\mathbf{X})$$

Fairness of the systemic risk allocation

Choosing $A = \rho_B(\mathbf{X})$, we obtain

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This means that by using $\mathbf{Q}_\mathbf{X}$ for valuation, the system utility maximization reduces to individual utility maximization problems for the banks without the "systemic" constraint $\mathbf{Y} \in \mathcal{C}$:

$$\forall n, \quad \sup_{Y^n} \left\{ \mathbb{E}\left[u_n(X^n + Y^n)\right] \mid \mathbb{E}_{Q^n_\mathbf{X}}[Y^n] = \mathbb{E}_{Q^n_\mathbf{X}}[Y^n_\mathbf{X}] \right\},$$

and therefore, **fairness from the point of view of individual banks**.

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In fact, the optimal allocation $\mathbf{Y}_\mathbf{X} = (Y^1_\mathbf{X}, \cdots, Y^N_\mathbf{X})$, is a **Pareto equilibrium**, as shown in the **SORTE** forthcoming paper:
Systemic Optimal Risk Transfer Equilibrium

Dual representation in the Grouping Example

For a partition **n** and for $\mathcal{C}^{(\mathbf{n})}$:

$$\rho(\mathbf{X}) = \max_{\mathbf{Q}\in\mathcal{D}} \left\{ \sum_{m=1}^{h} \mathbb{E}_{Q^m}[-\overline{X}_m] - \alpha_B(\mathbf{Q}) \right\},$$

with

$$\mathcal{D} := dom(\alpha_B) \cap \left\{ \frac{d\mathbf{Q}}{d\mathbb{P}} \in L_+^{\Phi^*} \mid Q^i = Q^j := Q^m \ \forall i,j \in I_m,\ \ Q^m(\Omega) = 1 \right\}$$

and

$$\overline{X}_m := \sum_{k\in I_m} X^k.$$

Exponential Setting

- $\mathcal{C} = \mathcal{C}^{(h)} \subseteq \mathcal{C}_{\mathbb{R}}$, for a partition of $\{1, ..., N\}$ in h groups.
- $u_n(x) = -e^{-\alpha_n x}$, $\alpha_n > 0$, $B < 0 = \sum_{n=1}^{N} u_n(+\infty)$.

$$\phi_n(x) := -u_n(-|x|) + u_n(0) = e^{\alpha_n |x|} - 1$$

$$M^{\phi_n} = M^{\exp} := \left\{ X \in L^0(\mathbb{R}) \mid \mathbb{E}[e^{c|X|}] < +\infty \text{ for all } c > 0 \right\},$$

$$\alpha_B(\mathbf{Q}) = \sum_{n=1}^{N} \mathbb{E}\left[\frac{dQ^n}{d\mathbb{P}} v_n' \left(\widehat{\lambda} \frac{dQ^n}{d\mathbb{P}} \right) \right] = \sum_{n=1}^{N} \frac{1}{\alpha_n} \left(H\left(Q^n, \mathbb{P}\right) + \ln\left(-\frac{B}{\beta \alpha_n} \right) \right)$$

with $\beta := \sum_{n=1}^{N} \frac{1}{\alpha_n}$ and $H\left(Q^n, \mathbb{P}\right) := \mathbb{E}\left[\frac{dQ^n}{d\mathbb{P}} \ln\left(\frac{dQ^n}{d\mathbb{P}} \right) \right]$.

In the exponential setting

- Existence, uniqueness and explicit formula for the optimal solution $\mathbf{Y_X} \in M^{\exp}$ to $\rho_B(\mathbf{X})$, $\rho_B^{\mathbf{Q_X}}(\mathbf{X})$, $\pi_A(\mathbf{X})$ and $\pi_A^{\mathbf{Q_X}}(\mathbf{X})$ (for $A := \rho_B(\mathbf{X})$)

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- Existence, uniqueness and explicit formula for the optimal solution $\mathbf{Q_X}$ to the dual problem associated to $\rho_B(\mathbf{X})$:

$$\frac{dQ^m_{\mathbf{X}}}{d\mathbb{P}} := \frac{e^{-\frac{1}{\beta_m}\overline{X}_m}}{\mathbb{E}\left[e^{-\frac{1}{\beta_m}\overline{X}_m}\right]} \quad m = 1, \cdots, h.$$

with $\beta_m = \sum_{k \in I_m} \frac{1}{\alpha_k}$, $\overline{X}_m = \sum_{k \in I_m} X^k$.

Take home message

Fairness is addressed by solving the dual problem

See the paper
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Thanks for your attention