Conditional Value-at-Risk via Copulas

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The general framework

Given

- risk factors $\mathbf{X} = (X_1, \ldots, X_d) \sim F_{\mathbf{X}}$, where

$$F_{\mathbf{X}}(\mathbf{x}) = \mathbb{P}\left(X_1 \leq x_1, \ldots, X_d \leq x_d\right);$$

- a financial position $\psi(\mathbf{X})$;
- a risk measure ρ;

the goal is

$$\text{calculate } \rho(\psi(\mathbf{X})).$$

Warning: $\rho(\psi(\mathbf{X}))$ only depends on the *joint* pdf $F_{\mathbf{X}}$ of $\mathbf{X}$.

Note: if time matters, one can consider the process $(\mathbf{X}_t)_{t=1,\ldots,T}$.

Current practice (?)

Given some risk factors $\mathbf{X} = (X_1, \ldots, X_d)$, we proceed as follow:

- Estimate the marginal d.f. F_i of each X_i, i.e.

$$F_i(x) = \mathbb{P}(X_i \leq x).$$

- Find a copula C such that

$$\mathbf{X} \sim F_{\mathbf{X}} = C(F_1, \ldots, F_d).$$

- Calculate $\rho(\psi(\mathbf{X}))$ either analytically or by means of a Monte-Carlo simulation from the joint d.f. $F_{\mathbf{X}}$.

Sklar's Theorem

Definition

For every $d \geq 2$, a d–dimensional copula (shortly, a d–copula) C is a d–dimensional distribution function whose univariate marginals are uniformly distributed on $[0, 1]$.

Theorem (Sklar, 1959)

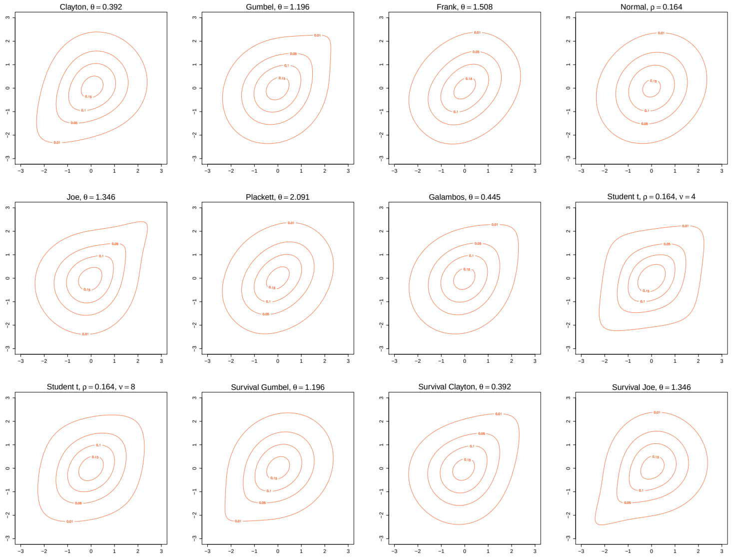
Let $(X_1, \ldots, X_d)$ *be a r.v. with continuous joint d.f.* F *and univariate marginals* F_1, F_2,..., F_d. *Then there exists a unique copula* C, *such that, for all* $\mathbf{x} \in \mathbb{R}^d$,

$$F(x_1, x_2, \ldots, x_d) = C\left(F_1(x_1), F_2(x_2), \ldots, F_d(x_d)\right).$$

C *is the d.f. of* $(F_1(X_1), \ldots, F_d(X_d))$ *and it equals*

$$C(u_1, \ldots, u_d) = F\left(F_1^{(-1)}(u_1), \ldots, F_d^{(-1)}(u_d)\right).$$

Example: the shape of copula models



Tail dependence

The notion of tail dependence is related to the comovement of two r.v.'s X and Y in the tails of their joint distribution. It makes mathematically precise statements like

given X is extreme, what is the
conditional probability of Y being also extreme?

Examples:

- In asset management, we are interested whether the drop of one (or more) stocks may influence the behavior of the other stocks in the portfolio (e.g., does diversification matter?).
- In credit portfolios, we are interested whether the default of a firm may increase or not the probability of default of other firms.
- In environmental science, we are interested about the occurrence of extreme events at multiple sites (e.g., flood risk maps).

Tail dependence coefficients

Let X and Y be continuous r.v.'s with d.f.'s F_X and F_Y, respectively, and copula C.
The upper tail dependence coefficient λ_U of (X, Y) is defined by

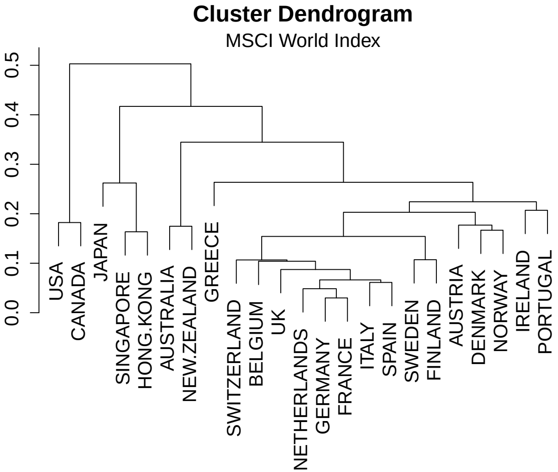
$$\lambda_U = \lim_{t \to 1^-} \mathbb{P}\left(Y > F_Y^{(-1)}(t) \mid X > F_X^{(-1)}(t)\right) = \lim_{t \to 1^-} \frac{1 - 2t + C(t,t)}{1-t};$$

and the lower tail dependence coefficient λ_L of (X, Y) is defined by

$$\lambda_L = \lim_{t \to 0^+} \mathbb{P}\left(Y \leq F_Y^{(-1)}(t) \mid X \leq F_X^{(-1)}(t)\right) = \lim_{t \to 0^+} \frac{C(t,t)}{t}$$

provided that the above limits exist.

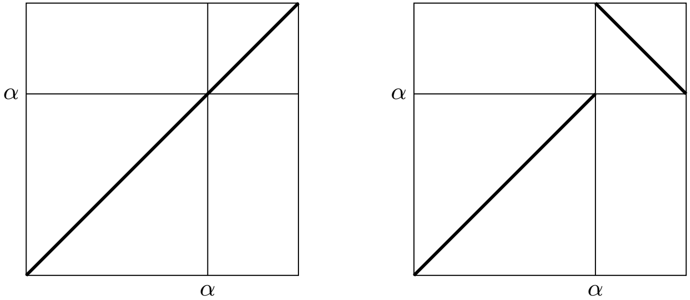
An application to financial time series



Tail-dependence based hierarchical clustering for the MSCI World Index Data according to complete linkage. Source: Morgan Stanley Capital International (MSCI) Developed Markets Index: daily observations from 2002-06-04 to 2010-06-10). For more details, see (D., Fernández-Sánchez, Pappadà, 2015).

Illustration: worst-case VaR_α copula for $d = 2$

Let L_1, L_2 be random losses whose dependence is represented by the comonotonicity copula M (left) and the copula C (right).

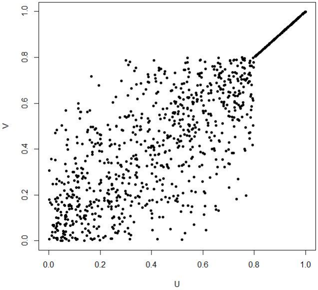


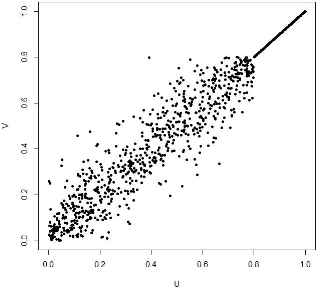
$$\mathrm{VaR}_\alpha(L_1) + \mathrm{VaR}_\alpha(L_2) \quad \leq \quad \sup_{L_1, L_2} \mathrm{VaR}_\alpha(L_1 + L_2)$$

(Makarov, 1981; Rueschendorf, 1982)

Illustration: upper comonotonicity

Let L_1, L_2 be random losses whose dependence is represented by the patchwork copulas C_1 (left) and C_2 (right).





$$\mathrm{VaR}_{\alpha}^{C_1}(L_1 + L_2) \quad = \quad \mathrm{VaR}_{\alpha}^{C_2}(L_1 + L_2)$$

In the plots, we visualize random sample of 1000 realizations from the copula $\langle B, C_B \rangle^{M_2}$ where $B = [0, 0.8]^2$, C_B is a Frank with Kendall's tau equal to: 0.5 (left) and 0.75 (right). For more details, see (D., Fernández–Sánchez and Sempi, 2013).

Conditional Value-at-Risk

VaR focuses on the risk of an individual institution *in isolation*. However, a single institution's risk measure does not necessarily reflect its connection to overall systemic risk. Some institutions are individually systemic – they are so interconnected and large that they can generate negative risk spillover effects on others.

(Adrian and Brunnermeier, AER, 2016)

Given two r.v.'s X and Y, the Conditional Value–at–Risk (CoVaR, for short) of Y given X can be generally defined by

$$\mathrm{CoVaR}^E(Y \mid X) = \mathrm{VaR}_\beta(Y \mid X \in E),$$

where E is a Borel set of the real line and $\beta \in (0,1)$. Usually, E represents the loss of X being at or above its VaR level.

(Adrian and Brunnermeier, AER, 2016)

Conditional Value-at-Risk and copulas

Let X and Y be profit/loss r.v.'s with continuous joint d.f. F, which can be expressed as $F = C(F_X, F_Y)$.

For $\alpha, \beta \in (0,1)$, we set

$$\mathrm{CoVaR}^{=}_{\alpha,\beta}(Y \mid X) = \mathrm{VaR}_{\beta}(Y \mid X = -\mathrm{VaR}_{\alpha}(X)).$$

Since C coincides with the joint d.f. of $(F_X(X), F_Y(Y)) = (U, V)$, then

$$\mathrm{CoVaR}^{=}_{\alpha,\beta}(Y \mid X) = \mathrm{VaR}_{v_*}(Y),$$

where $v_* = v_*(\alpha, \beta, C)$ is computed via

$$v_* = \inf\{v \in [0,1] \colon F_{V|U=\alpha}(v) > \beta\}. \tag{1}$$

Conditional Value-at-Risk and copulas

When C is continuously differentiable, (1) can be rewritten as

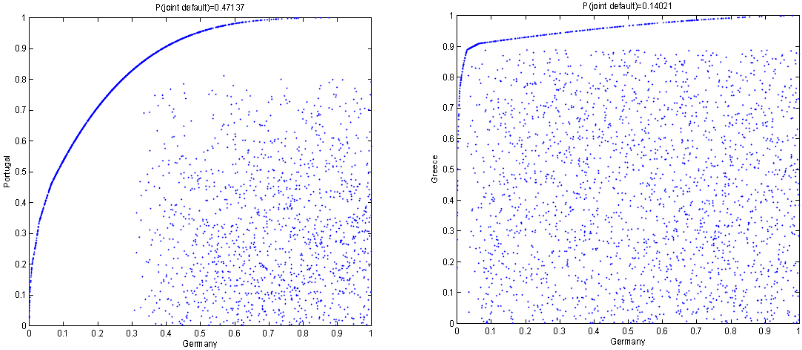
$$v_* = \inf\{v \in [0,1] \colon \partial_1 C(\alpha, v) > \beta\}.$$

Moreover, $\mathrm{CoVaR}^{=}_{\alpha,\beta}(Y \mid X)$ fulfills

$$\partial_1 C(\alpha, F_Y(-\mathrm{CoVaR}^{=}_{\alpha,\beta}(Y \mid X))) = \beta.$$

However, C ma not have first order partial derivatives everywhere!
Example: $(X, Y) \sim C(F_X, F_Y)$, and $\mathbb{P}(X = \varphi(Y)) > 0$.

Copulas with a singular component: A Eurozone case study



Maximal probability of joint defaults induced by CDS spread for Germany vs Portugal (left) and Germany vs Greece (right) on December 12, 2012. Courtesy of J.F. Mai. For more details, see (Mai and Scherer, 2014).

Conditional Value-at-Risk and copulas

To provide a more general definition of $\text{CoVaR}^{=}$, we consider the left–sided upper Dini derivative of C with respect to the first coordinate. Specifically, for every $u \in (0,1]$ and $v \in [0,1]$, we set

$$D_1 C(u,v) = \limsup_{h \to 0^+} \frac{C(u,v) - C(u-h,v)}{h}.$$

It is easy to show that, for every $v \in [0,1]$, $K_C(u,[0,v]) = D_1 C(u,v)$ for almost all $u \in [0,1]$, where K_C is a version of the conditional distribution of V given U, also known as *Markov kernel of* C.

Therefore, in order to calculate $\text{CoVaR}^{=}$, we propose to use

$$v_* = \inf\{v : D_1\, C(\alpha, v) > \beta\}.$$

(Bernardi, D. and Jaworski, 2017)

Example

- For the independence copula $\Pi_2(u,v) = uv$

$$v_*(\alpha, \beta, \Pi_2) = \beta$$

- For the comonotonicity copula $M_2(u,v) = \min(u,v)$

$$v_*(\alpha, \beta, M_2) = \alpha.$$

- For the countermonotonicity copula $W_2(u,v) = \max(u+v-1, 0)$

$$v_*(\alpha, \beta, W_2) = 1 - \alpha.$$

In particular, for $\alpha = \beta$, if $(X, Y) \sim \Pi_2(F, G)$, $(X', Y') \sim M_2(F, G)$, then

$$\text{CoVaR}^{=}_{\alpha,\alpha}(Y \mid X) = \text{CoVaR}^{=}_{\alpha,\alpha}(Y' \mid X'). \quad ☹$$

Modified Conditional Value-at-Risk

For $\alpha, \beta \in (0,1)$, we set

$$\mathrm{CoVaR}^{\leq}_{\alpha,\beta}(Y|X) = \mathrm{VaR}_{\beta}(Y \mid X \leq -\mathrm{VaR}_{\alpha}(X)).$$

If the continuous joint d.f. F of the random pair (X, Y) is expressed as $F = C(F_X, F_Y)$, then

$$\mathrm{CoVaR}^{\leq}_{\alpha,\beta}(Y \mid X) = \mathrm{VaR}_{w_*}(Y),$$

where w_* solves the equation $C(\alpha, w_*) = \alpha\beta$.

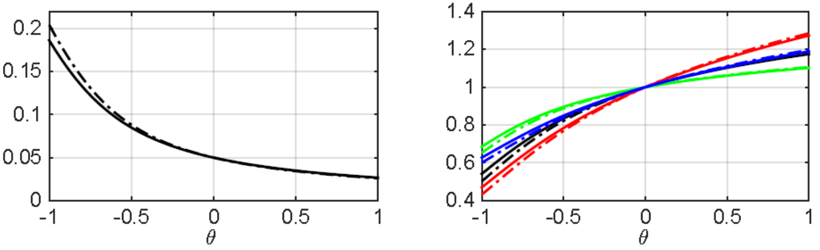
(Girard and Ergün, 2013)

If $(X, Y) \sim C(F_X, F_Y)$ and $(X', Y') \sim C'(F_{X'}, F_{Y'})$ with continuous $F_X, F_{X'}, F_Y = F_{Y'}$, then $C \leq C'$ implies

$$\mathrm{CoVaR}^{\leq}_{\alpha,\beta}(Y \mid X) \leq \mathrm{CoVaR}^{\leq}_{\alpha,\beta}(Y' \mid X').$$

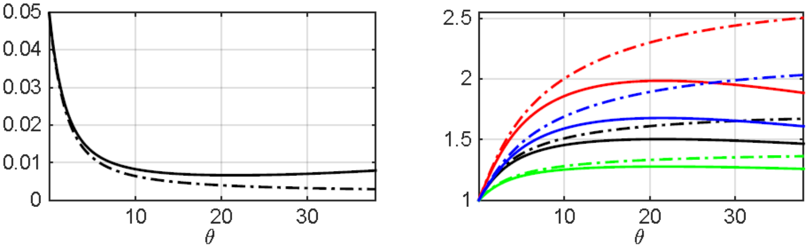
(Mainik and Schaanning, 2015)

Example: EFGM copulas



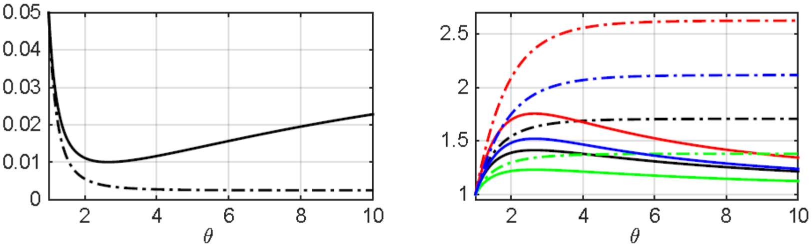
(Left panel): Plot of $v_*(\alpha, \beta, C_\theta^{\mathbf{EFGM}})$ *(black line)* and $w_*(\alpha, \beta, C_\theta^{\mathbf{EFGM}})$ *(black dotted line)* for different θ values. *(Right panel)*: $\mathrm{CoVaR}^{=}_{\alpha,\beta}(Y \mid X)/\mathrm{VaR}_\beta(Y)$ *(continuous lines)* and $\mathrm{CoVaR}^{\leq}_{\alpha,\beta}(Y \mid X)/\mathrm{VaR}_\beta(Y)$ *(dotted lines)*, for a random pair $(X, Y) \sim C_\theta^{\mathbf{EFGM}}(F_X, F_Y)$ with different marginals, namely Gaussian $\mathsf{N}(0, 1)$ *(black)*, Student–t $\mathsf{T}_\nu(0, 1)$ with $\nu = 4$ *(red)*, Skew–Normal $\mathsf{SN}_\lambda(0, 1)$ with $\lambda = 5$ *(green)* and Skew Student–t $\mathsf{ST}_{\lambda,\nu}(0, 1)$ with $\nu = 4$ and $\lambda = 5$ *(blue)*. Here, $\alpha = \beta = 0.05$. For more details, see (Bernardi, D. and Jaworski, 2017).

Example: Frank copulas



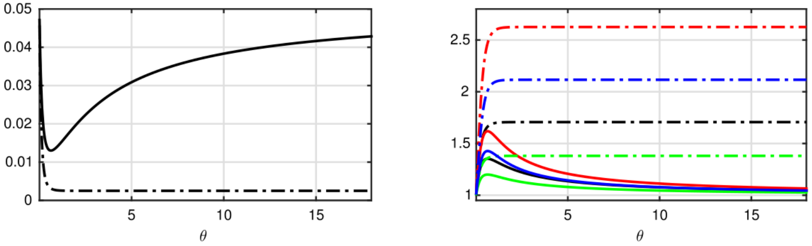
(Left panel): Plot of $v_*(\alpha, \beta, C_\theta^{\mathbf{Fr}})$ *(black line)* and $w_*(\alpha, \beta, C_\theta^{\mathbf{Fr}})$ *(black dotted line)* for different θ values.
(Right panel): $\mathrm{CoVaR}^{=}_{\alpha,\beta}(Y \mid X) / \mathrm{VaR}_\beta(Y)$ *(continuous lines)* and $\mathrm{CoVaR}^{\leq}_{\alpha,\beta}(Y \mid X) / \mathrm{VaR}_\beta(Y)$ *(dotted lines)*, for a random pair $(X, Y) \sim C_\theta^{\mathbf{Fr}}(F_X, F_Y)$ with different marginals, namely Gaussian $\mathsf{N}(0, 1)$ *(black)*, Student–t $\mathsf{T}_\nu(0, 1)$ with $\nu = 4$ *(red)*, Skew–Normal $\mathsf{SN}_\lambda(0, 1)$ with $\lambda = 5$ *(green)* and Skew Student–t $\mathsf{ST}_{\lambda,\nu}(0, 1)$ with $\nu = 4$ and $\lambda = 5$ *(blue)*. Here, $\alpha = \beta = 0.05$. For more details, see (Bernardi, D. and Jaworski, 2017).

Example: Gumbel copulas



(Left panel): Plot of $v_*\left(\alpha, \beta, C_\theta^{\mathbf{Gu}}\right)$ *(black line)* and $w_*\left(\alpha, \beta, C_\theta^{\mathbf{Gu}}\right)$ *(black dotted line)* for different θ values.
(Right panel): $\mathrm{CoVaR}^{=}_{\alpha,\beta}(Y \mid X)/\mathrm{VaR}_\beta(Y)$ *(continuous lines)* and $\mathrm{CoVaR}^{\leq}_{\alpha,\beta}(Y \mid X)/\mathrm{VaR}_\beta(Y)$ *(dotted lines)*, for a random pair $(X, Y) \sim C_\theta^{\mathbf{Gu}}(F_X, F_Y)$ with different marginals, namely Gaussian $\mathsf{N}(0, 1)$ *(black)*, Student–t $\mathsf{T}_\nu(0, 1)$ with $\nu = 4$ *(red)*, Skew–Normal $\mathsf{SN}_\lambda(0, 1)$ with $\lambda = 5$ *(green)* and Skew Student–t $\mathsf{ST}_{\lambda,\nu}(0, 1)$ with $\nu = 4$ and $\lambda = 5$ *(blue)*. Here, $\alpha = \beta = 0.05$. For more details, see (Bernardi, D. and Jaworski, 2017).

Example: Clayton copulas

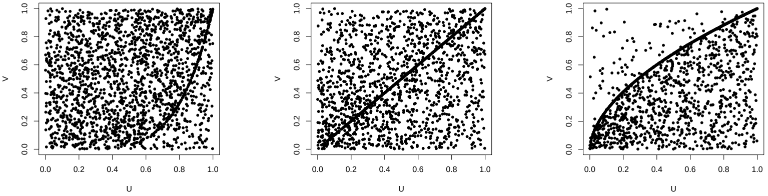


(Left panel): Plot of $v_*(\alpha, \beta, C_\theta^{\mathbf{MTC}})$ *(black line)* and $w_*(\alpha, \beta, C_\theta^{\mathbf{MTC}})$ *(black dotted line)* for different θ values.
(Right panel): $\mathrm{CoVaR}_{\alpha,\beta}(Y \mid X) / \mathrm{VaR}_\beta(Y)$ *(continuous lines)* and $\mathrm{CoVaR}^{\leq}_{\alpha,\beta}(Y \mid X) / \mathrm{VaR}_\beta(Y)$ *(dotted lines)*, for a random pair $(X, Y) \sim C_\theta^{\mathbf{MTC}}(F_X, F_Y)$ with different marginals, namely Gaussian $\mathsf{N}(0, 1)$ *(black)*, Student–t $\mathsf{T}_\nu(0, 1)$ with $\nu = 4$ *(red)*, Skew–Normal $\mathsf{SN}_\lambda(0, 1)$ with $\lambda = 5$ *(green)* and Skew Student–t $\mathsf{ST}_{\lambda,\nu}(0, 1)$ with $\nu = 4$ and $\lambda = 5$ *(blue)*. Here, $\alpha = \beta = 0.05$. For more details, see (Bernardi, D. and Jaworski, 2017).

Example: Marshall–Olkin copulas

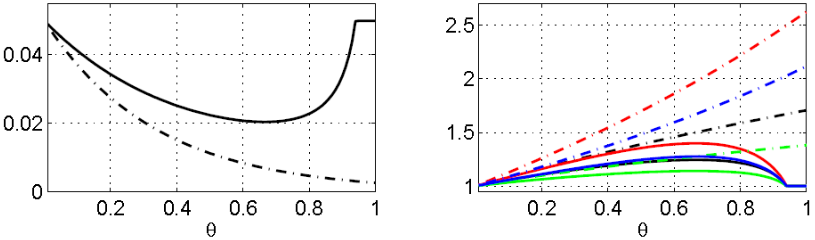
For $a, b \in (0,1)$ the Marshall–Olkin copula is defined as

$$C^{\mathbf{MO}}_{a,b}(u,v) = \begin{cases} u^{1-a}v, & u^a \geq v^b, \\ uv^{1-b}, & u^a < v^b. \end{cases}$$



Random sample of 2000 realizations from the Marshall–Olkin copula with parameters $(0.5, 0.1)$ (left), $(0.5, 0.5)$ (center), and $(0.5, 0.9)$ (right).

Example: Marshall–Olkin copulas



(Left panel): Plot of $v_*(\alpha, \beta, C_\theta^{\mathbf{MO}})$ *(black line)* and $w_*(\alpha, \beta, C_\theta^{\mathbf{MO}})$ *(black dotted line)* for $\theta = a = b$.

(Right panel): $\mathrm{CoVaR}^{=}_{\alpha,\beta}(Y \mid X)/\mathrm{VaR}_\beta(Y)$ *(continuous lines)* and $\mathrm{CoVaR}^{\leq}_{\alpha,\beta}(Y \mid X)/\mathrm{VaR}_\beta(Y)$ *(dotted lines)*, for a random pair $(X, Y) \sim C_\theta^{\mathbf{MO}}(F_X, F_Y)$ with different marginals, namely Gaussian $\mathsf{N}(0,1)$ *(black)*, Student–t $\mathsf{T}_\nu(0,1)$ with $\nu = 4$ *(red)*, Skew–Normal $\mathsf{SN}_\lambda(0,1)$ with $\lambda = 5$ *(green)* and Skew Student–t $\mathsf{ST}_{\lambda,\nu}(0,1)$ with $\nu = 4$ and $\lambda = 5$ *(blue)*. Here, $\alpha = \beta = 0.05$. For more details, see (Bernardi, D. and Jaworski, 2017).

Example: Marshall–Olkin copulas

Given $a, b \in (0,1)$, for the Marshall–Olkin copula $C^{\mathbf{MO}}_{a,b}$, it follows

$$v_*(\alpha,\beta,C^{\mathbf{MO}}_{a,b}) = \begin{cases} \frac{\beta\alpha^a}{1-a}, & 0<\beta<(1-a)\alpha^{(1-b)a/b}, \\ \alpha^{a/b}, & (1-a)\alpha^{(1-b)a/b} \le \beta \le \alpha^{(1-b)a/b}, \\ \beta^{1/(1-b)}, & \alpha^{(1-b)a/b} < \beta < 1, \end{cases}$$

and

$$w_*(\alpha,\beta,C^{\mathbf{MO}}_{a,b}) = \begin{cases} \beta\alpha^a, & 0<\beta\le\alpha^{(1-b)a/b}, \\ \beta^{1/(1-b)}, & \alpha^{(1-b)a/b} < \beta < 1. \end{cases}$$

In particular, it is interesting to note that

$$\lim_{\alpha\to 0^+} v_*(\alpha,\beta,C^{\mathbf{MO}}_{a,b}) = \beta^{\frac{1}{1-b}} = \lim_{\alpha\to 0^+} w_*(\alpha,\beta,C^{\mathbf{MO}}_{a,b}).$$

(Bernardi, D. and Jaworski, 2017)

Example: Archimedean copulas

For a (strict) Archimedean copula

$$C_{\varphi,\psi}(u,v) = \psi(\varphi(u)+\varphi(v))$$

with $\varphi(0) = +\infty$, and φ regularly varying at 0 with a negative index, i.e. $\lim_{t\to 0^+} \frac{\varphi(tx)}{\varphi(t)} = x^{-d}$,

$$\lim_{\alpha\to 0^+} v_*(\alpha,\beta,C_{\varphi,\psi}) = \lim_{\alpha\to 0^+} w_*(\alpha,\beta,C_{\varphi,\psi}) = 0$$

with

$$\lim_{\alpha\to 0^+} \frac{v_*(\alpha,\beta,C_{\varphi,\psi})}{\alpha} = \left(\beta^{-d/(d+1)} - 1\right)^{-1/d},$$

$$\lim_{\alpha\to 0^+} \frac{w_*(\alpha,\beta,C_{\varphi,\psi})}{\alpha} = \left(\beta^{-d} - 1\right)^{-1/d}.$$

(Bernardi, D. and Jaworski, 2017)

Multivariate Conditional Value-at-Risk

Let $(\mathbf{X}, Y)$ be a random vector. Let $\mathcal{S}$ be an *upper*[1] Borel set in $\mathbb{R}^d$, which is interpreted as an hazard scenario.

The CoVaR of Y given that $\mathbf{X} \in \mathcal{S}$ is defined as

$$\text{CoVaR}^{\mathcal{S}}_{\alpha,\beta}(Y \mid \mathbf{X}) = \text{VaR}_{\beta}(Y \mid \mathbf{X} \in \mathcal{S}),$$

where $\beta \in (0,1)$ and $\mathbb{P}(\mathbf{X} \in \mathcal{S}) = 1 - \alpha \in (0,1)$.

It can be easily seen that

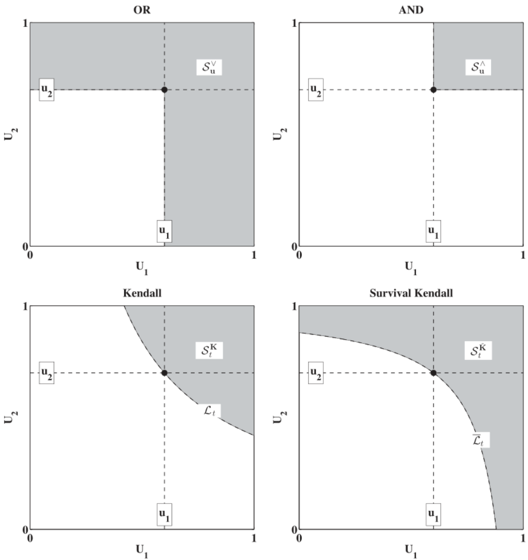
$$\overline{\mathcal{S}} = \{\mathbf{z} \in \mathbb{R}^d_+ : \psi(\mathbf{z}) \geq 1\},$$

for a continuous and increasing function ψ.

(Bernardi et al., 2018)

[1] If $\mathcal{S}$ is an upper set, then, for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$, $\mathbf{x} \in \mathcal{S}$ and $\mathbf{y} \geq \mathbf{x}$ (component-wise) imply $\mathbf{y} \in \mathcal{S}$.

Example: hazard scenarios



Multivariate Conditional Value-at-Risk

Given $\alpha \in (0, 1)$ such that $\mathbb{P}(\mathbf{X} \in \mathcal{S}) = 1 - \alpha$, for all y we have

$$\mathbb{P}(Y \geq y \mid \mathbf{X} \in \mathcal{S}) = \frac{\widehat{D}(1 - \alpha, \overline{G}(y))}{1 - \alpha},$$

where $Y \sim (1 - \overline{G})$ and $\widehat{D}$ is the bivariate survival copula associated with $(\psi_{\mathbf{x}}(\mathbf{X}), Y)$.

Thus,

$$\mathrm{CoVaR}_{\mathcal{S}_{\mathbf{x}},\beta}(Y \mid \mathbf{X}) = \overline{G}^{-1}\left((h_{1-\alpha}^{\widehat{D}})^{-1}((1 - \alpha)(1 - \beta))\right),$$

where $h_{1-\alpha}^{\widehat{D}}(t) = \widehat{D}(1 - \alpha, t)$ is the section of the copula $\widehat{D}$ at the point $1 - \alpha$, having range $[0, 1 - \alpha]$.

(Bernardi et al., 2018)

CoVaR under AND scenario

Here, we are interested in the calculation of the conditional risk when the conditioning event is an AND HS of type $\{\mathbf{X} \geq \mathbf{x}\}$. Then

$$\mathrm{CoVaR}_{\alpha,\beta}\left(Y \mid \mathbf{X} \geq \mathbf{x}\right) = \overline{G}^{-1}\left(\left(h_{\mathbf{u}}^{\widehat{C}}\right)^{-1}\left(\left(1-\beta\right)\left(1-\alpha\right)\right)\right),$$

where $\overline{G}$ is the survival function of Y,

$$h_{\mathbf{u}}^{\widehat{C}}\left(\cdot\right) = \widehat{C}\left(u_1, u_2, \ldots, u_d, \cdot\right),$$

is the section of $\widehat{C}$, the survival copula of $(\mathbf{X}, Y)$, with respect to the $(d+1)$–component.

(Bernardi et al., 2018)

CoVaR under AND scenario: remark

- $\text{CoVaR}_{\alpha,\beta}(Y \mid \mathbf{X} \geq \mathbf{x})$ does not change when $\mathbf{x}$ is replaced by any other point lying on the level curve

$$\{\mathbf{y} \in \mathbb{R}^d \colon F(\mathbf{x}) = 1 - \alpha\}.$$

- Let $(\mathbf{X}, Y)$ and $(\mathbf{X}', Y')$ be random vectors with survival copulas C and C', respectively, and identical continuous marginal survival functions. If $C \geq C'$ (PLOD order), then

$$\text{CoVaR}_{\alpha,\beta}(Y \mid \mathbf{X} \geq \mathbf{x}) \geq \text{CoVaR}_{\alpha,\beta}\left(Y' \mid \mathbf{X}' \geq \mathbf{x}'\right) \qquad ☺$$

for $\overline{F}_{\mathbf{X}}(\mathbf{x}) = 1 - \alpha$ and $\overline{F}_{\mathbf{X}'}(\mathbf{x}') = 1 - \alpha$, with $\mathbf{x}' \leq \mathbf{x}$.

(Bernardi et al., 2018)

CoVaR under AND scenario: example

Suppose that $(\mathbf{X}, Y) \sim \widehat{C}(\overline{F_1}, \ldots, \overline{F_d}, \overline{G})$ for a survival copula $\widehat{C}$ of Archimedean type, i.e. such that

$$\widehat{C}(\mathbf{u}) = \varphi^{-1}(\varphi(u_1) + \cdots + \varphi(u_{d+1}))$$

for a suitable strictly decreasing generator φ, with $\varphi(0) = +\infty$.

Thus,

$$\mathrm{CoVaR}_{\alpha,\beta}(Y \mid \mathbf{X} \geq \mathbf{x}) = \overline{G}^{-1}\left(\varphi^{-1}\left[\varphi((1-\beta)(1-\alpha)) - \varphi(1-\alpha)\right]\right),$$

and it does not explicitly depend on the dimension d.

(Bernardi et al., 2018)

An environmental illustration

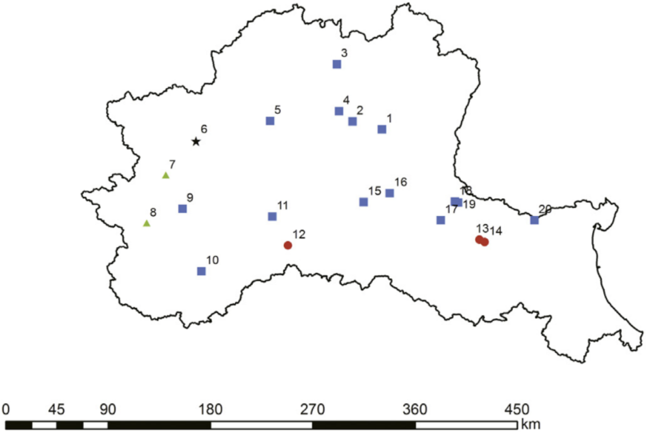
In order to illustrate the practical application of the presented methodology, we describe how flood risks are interconnected in a region.

We consider a set of three certified gauge stations recording annual maximum flood data in the following sites:

- Airole - Piena,
- Merelli - Centrale Argentina,
- Ponte Poggi - Eller.

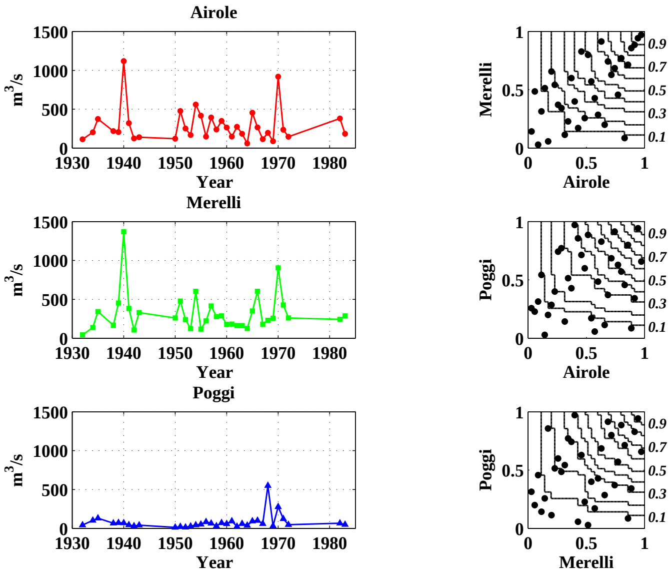
> [...] the flood risk management should require the implementation of suitable flood hazard maps covering the geographical areas which could be flooded according to the following scenarios: (a) floods with a low probability, or extreme event scenarios; (b) floods with a medium probability (likely return period $\geq$ 100 years); (c) floods with a high probability, where appropriate.
>
> (The European Parliament and The Council, 2007, p. 30, chap. III, Article 6.3)

Flood risk map: an example



(Pappadà et al., 2018)

An environmental illustration



An environmental illustration

The following trivariate copula can be used for modeling the dependence among the three stations:

$$C(u_1, u_2, u_3) = u_1^{1-b_1} u_2^{1-b_2} u_3^{1-b_3} \min(u_1^{b_1}, u_2^{b_2}, u_3^{b_3}),$$

where $b_1 = 0.5921$, $b_2 = 0.6862$, and $b_3 = 0.2349$.
This copula is non-exchangeable and may model the differences in pairwise positive dependence among the series.

(Bernardi et al., 2018)

An environmental illustration

	$\mathrm{VaR}_\alpha(X_A)$	$\mathrm{VaR}_\alpha(X_M)$	$\mathrm{VaR}_\alpha(X_P)$
$\alpha = 0.90$	720.0415	755.9679	187.5132
$\alpha = 0.95$	1036.028	1048.431	276.9017
$\alpha = 0.99$	2101.462	1947.033	603.5813

VaR_α related to annual maximum flood data (m^3/s) in Airole (X_A), Merelli (X_M), and Poggi (X_P), for different values of α.

	$R_\alpha^\wedge(X_A \mid X_M, X_P)$	$R_\alpha^\wedge(X_M \mid X_A, X_P)$	$R_\alpha^\wedge(X_P \mid X_A, X_M)$
$\alpha = 0.90$	2.7086	2.4541	2.9346
$\alpha = 0.95$	3.3245	2.8509	3.8203
$\alpha = 0.99$	5.3146	4.0124	6.8517

$R_\alpha^\wedge(Y \mid X_1, X_2) = \mathrm{CoVaR}_{\alpha,\alpha}(Y \mid X_1 \geq x_1, X_2 \geq x_2)/\mathrm{VaR}_\alpha(Y)$ related to annual maximum flood data (m^3/s) in Airole (X_A), Merelli (X_M), and Poggi (X_P), for different values of α.

Questions? Comments?

Thanks for your attention!

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