Financial applications of expectiles

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Outline

- Expectiles: definition, financial interpretation and main properties
- Expectiles as risk measures
- Backtesting with scoring functions
- Measuring implied volatility with implicit expectiles
- Risk parity portfolios with expectiles

Expectiles

The expectiles have been introduced in the statistical literature by Newey and Powell (1987) as the minimizers of a piecewise quadratic expected loss:

$$e_\alpha(X) = \arg\min_{x \in \mathbb{R}} \mathbb{E}[\alpha(X-x)_+^2 + (1-\alpha)(X-x)_-^2], \tag{D1}$$

with $X \in L^2$ and $\alpha \in (0,1)$. Expectiles can be seen as:

- a one-parameter asymmetric generalization of the mean, that corresponds to the case $\alpha = 1/2$
- a quadratic analogue of the usual quantiles, that are the minimizers of a piecewise linear expected loss:

$$[q_\alpha^-(X), q_\alpha^+(X)] = \arg\min_{x \in \mathbb{R}} \mathbb{E}[\alpha(X-x)_+ + (1-\alpha)(X-x)_-].$$

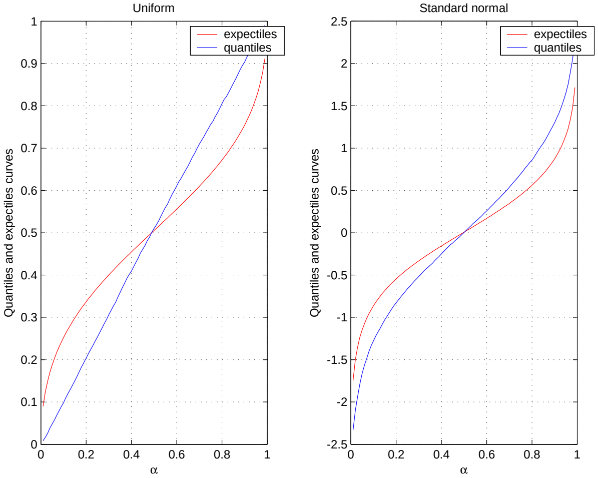
Expectiles

In contrast with the case of quantiles, problem (D1) has always a unique solution, identified by the first order condition

$$\alpha\mathbb{E}[(X - e_\alpha)_+] = (1-\alpha)\mathbb{E}[(X - e_\alpha)_-], \tag{D2}$$

which is indeed an alternative and better definition than (D1) since it is well posed for each $X \in L^1$, the natural domain of expectiles.
For the most common distributions, the expectiles e_α are closer to the center of the distribution than the corresponding quantiles q_α, and the expectile curve and the quantile curve intersect in a single $\overline{\alpha}$.

Expectiles and quantiles



Properties of Expectiles

We recall the main properties of expectiles from Nevey and Powell (1987) and Bellini et al. (2014). Let $X \in L^1$ and $\alpha \in (0,1)$. Then:

- $e_\alpha(X+h) = e_\alpha(X) + h$, for each $h \in \mathbb{R}$
- $e_\alpha(\lambda X) = \lambda e_\alpha(X)$, for each $\lambda \geq 0$
- $X \leq Y$ a.s., $P(X < Y) > 0 \Rightarrow e_\alpha(X) < e_\alpha(Y)$, for each $\alpha \in (0,1)$
- e_α is strictly increasing with respect to α
- e_α is continuous with respect to α
- $\lim_{\alpha \to 0^+} e_\alpha(X) = \operatorname{ess\,inf}(X)$, $\lim_{\alpha \to 1^-} e_\alpha(X) = \operatorname{ess\,sup}(X)$
- If $\alpha \leq 1/2$, then $e_\alpha(X+Y) \geq e_\alpha(X) + e_\alpha(Y)$; if $\alpha \geq 1/2$ then $e_\alpha(X+Y) \leq e_\alpha(X) + e_\alpha(Y)$.
- $e_\alpha(-X) = -e_{1-\alpha}(X)$.

Axiomatization of expectiles

Expectiles with $\alpha \geq 1/2$ have been axiomatized as the only elicitable coherent risk measures. Recall that a statistical functional T is said to be elicitable if it can be defined as the minimizer of the expected value of a suitable consistent scoring function:

$$T(F) = \arg\min_{x \in \mathbb{R}} \int S(x, y) dF(y).$$

A necessary condition for elicitability is the convexity of the level sets with respect to mixtures (the so called CxLS property):

$$T(F) = T(G) = \gamma \Rightarrow T(\lambda F + (1 - \lambda) G) = \gamma,$$

indeed expectiles with $\alpha \geq 1/2$ can equivalently be axiomatized as the only coherent risk measures with the CxLS property (see Bellini et al., 2014, Delbaen et al., 2016, and the references therein).

Financial interpretation of expectiles

In order to provide a simple financial interpretation of expectiles, we define the Expectile VaR as

$$EVaR_\alpha(X) = -e_\alpha(X), \text{ for } \alpha \in (0, 1/2),$$

paralleling the standard definition $VaR_\alpha(X) = -q_\alpha(X)$.
With this sign convention, the acceptance set of a risk measure ρ is

$$\mathcal{A}_\rho := \{X \in L^1 \mid \rho(X) \leq 0\},$$

and ρ can be recovered from $\mathcal{A}_\rho$ by the formula

$$\rho(X) = \inf\{m \in \mathbb{R} \mid X + m \in \mathcal{A}_\rho\}.$$

Financial interpretation of expectiles

In the case of $EVaR_\alpha$, the acceptance set is given by

$$\mathcal{A}_{EVaR_\alpha} = \left\{ X \in L^1 \;\middle|\; \frac{\mathbb{E}[X_+]}{\mathbb{E}[X_-]} \geq \frac{1-\alpha}{\alpha} \right\}.$$

A position is thus acceptable for $EVaR_\alpha$ if and only if its gain-loss ratio is higher than a fixed threshold $(1-\alpha)/\alpha \geq 1$.
For comparison, the acceptance set for VaR_α is

$$\mathcal{A}_{VaR_\alpha} = \left\{ X \;\middle|\; \mathbb{P}(X < 0) \leq \alpha \right\},$$

that can be equivalently written as

$$\mathcal{A}_{VaR_\alpha} = \left\{ X \;\middle|\; \frac{\mathbb{P}(X \geq 0)}{\mathbb{P}(X < 0)} \geq \frac{1-\alpha}{\alpha} \right\}.$$

Critiques to expectiles as risk measures

In my opinion, the main drawbacks underlined in the literature are:

- expectiles 'depend on the whole distribution' and not only on the left tail as the quantiles or the Conditional Value at Risk
- expectiles can be strictly subadditive also for comonotonic variables

These drawbacks have to be weighted with the advantages given by the coherence property and by the existence of consistent scoring functions:

- the possibilty of using scoring functions for traditional or comparative backtesting
- the possibility of using scoring functions for expectile forecasting by means of regression or more modern statistical techniques such as regression trees, random forests, etc. etc.

Backtesting with scoring functions

Letting $\hat{\rho}_k$ be a sequence of forecasts of a risk measure ρ, X_k the realized logreturn and S a consistent scoring function for ρ, the realized score is

$$\widehat{S}_n(x) = \frac{1}{n}\sum_{k=1}^{n} S(\hat{\rho}_k, X_k).$$

The model with the lower realized score is better. We distinguish between

- traditional backtesting, (see e.g. Lopez, 1999, Wong, 2010, Holzmann and Eulert, 2014, Bellini et al., 2019) where the null is related with the ability of a single model to correctly forecast the risk measure ρ
- comparative backtesting (see e.g. Fissler et al., 2016 and Ziegel and Nolde, 2016) where the null hypothesis is related to the comparative merit of two models

Backtesting with scoring functions

Comparative backtesting is based on a Diebold-Mariano test on the normalized difference of the realized scores of the two competing models. In order to perform traditional backtesting with the realized score it is necessary to derive the distribution of the realized score under the null hypothesis that the forecasting model is correct; the model is then rejected if the realized score is too big. Since an elicitable functional typically satisfies also a first order condition of the form

$$\mathbb{E}[I(\rho, X)] = 0$$

for a suitable identification function I, an alternative strategy is to backtest by means of the empirical identification function

$$\widehat{I}_n(x) = \frac{1}{n} \sum_{k=1}^{n} I(\hat{\rho}_k, X_k)$$

that should be close to 0 if the model is correctly specified.

An example on simulated data

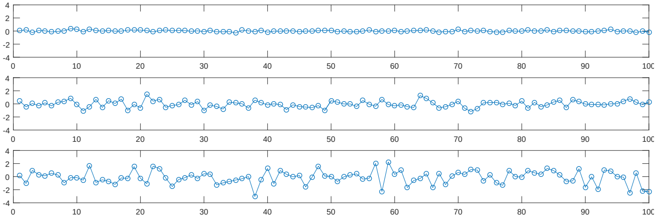
In Bellini et al. (2019) we compared the empirical power of the two backtesting strategies for VaR and expectiles forecasting on simulated data, with $X_i = N(\mu_i, 1)$, with X_i independent, $i = 1, ..., 100$, for three fixed vector of means μ_A, μ_B, μ_C, randomly chosen in advance.
The modeler wrongly believes that $X_i = N(0, 1)$, with X_i i.i.d, hence his VaR forecasts are constant and equal to $q_{0.05}(Y_i) = -1.6449$ and similarly its expectile forecasts are constant and equal to $e_{0.05}(Y) = -1.1402$.
For the case of VaR, we reject if the number of violations N_V satisfies $N_V \leq 1$ or $N_V \geq 10$, that corresponds to a level of the binomial test

$$P(N_V \leq 1) + P(N_V \geq 10) = 6,527\%.$$

The threshold $\overline{S} = 0.1216$ for the realized score $\widehat{S}$ is chosen to satisfy

$$P(\widehat{S} > \overline{S}) = 6,527\%.$$

An example on simulated data



Time-varying vectors of means μ^A, μ^B and μ^C.

An example on simulated data

In order to compare the empirical power of the two test we risimulate $N = 100000$ times models A, B and C.

In the case of VaR, the fraction of rejections are:

	$\mu = 0$	$\mu = \mu_A$	$\mu = \mu_B$	$\mu = \mu_C$
$R1$	$6,56\%$	$6,44\%$	$17,71\%$	$80,56\%$
$R2$	$6,66\%$	$6,98\%$	$41,76\%$	$97,60\%$

In the case of expectiles, we get

	$\mu = 0$	$\mu = \mu_A$	$\mu = \mu_B$	$\mu = \mu_C$
$R1$	$5,00\%$	$4,98\%$	$23,11\%$	$98,71\%$
$R2$	$5,01\%$	$5,59\%$	$37,00\%$	$99,87\%$

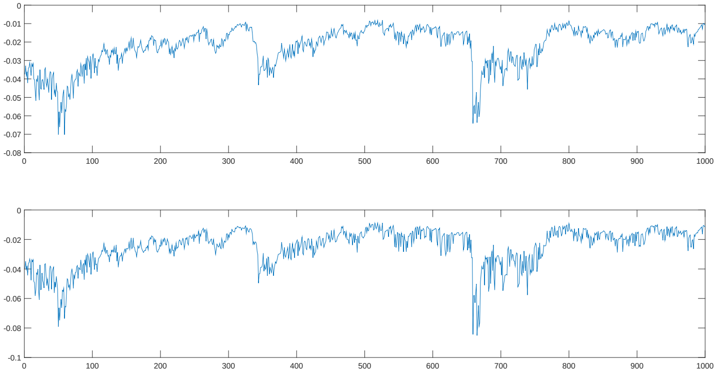
A real data example

In order to backtest with realized scores a more complex econometric models we suggest two possible approaches:

- to risimulate the empirical distribution of the realized score as e.g. in Bellini and Di Bernardino (2017)
- to first apply a probability integral transform $U_i = F_{Y_i}(Y_i)$ and then to use the asymptotic distribution of the realized score in the uniform i.i.d. case, similarly to Kerkhof and Melenberg (2004)

As an example we consider two AR(1)-Garch(1,1) models, with normal and t innovations, estimated on the daily logreturns of the SP500 Index from 03/01/2007 to 14/12/2012. Estimation has been performed on rolling windows of 500 logreturns, so we have 1000 distributional forecast. We backtest both models by means of the number of violations, the realized scores and the realized identification function.

A real data example



Forecasts of the 5%-VaR (upper part) and 1%-expectile (lower part) of the AR(1)-Garch(1,1) model with t innovations.

A real data example

	$\widehat{S}^{(v)}$	$\widehat{S}^{(e)}$	$\widehat{S}^{(v)}$	$\widehat{S}^{(e)}$
score	1.50×10^{-3}	1.38×10^{-5}	1.49×10^{-3}	1.28×10^{-5}
mean	1.31×10^{-3}	1.00×10^{-5}	1.52×10^{-3}	1.58×10^{-5}
std	4.95×10^{-5}	1.10×10^{-6}	8.06×10^{-5}	1.01×10^{-5}
p-val	0.00072	0.022	0.52	0.61

Realized values of the quantile and the expectiles scores $\widehat{S}^{(v)}$ and $\widehat{S}^{(e)}$ in the normal (left) and t (right) models, compared with their mean and standard deviation computed by risimulations.

Implicit expectiles

Our second financial application of expectiles (Bellini et al., 2018) is to extract the variability of the risk neutral distribution from option prices. Since the prices of european calls and puts as a function of the strike K are

$$C(K) = e^{-rT}\mathbb{E}_{\mathbb{Q}}[(S_T - K)_+] \text{ and } P(K) = e^{-rT}\mathbb{E}_{\mathbb{Q}}[(S_T - K)_-],$$

the equation (D2) defining expectiles can be rewritten as

$$\alpha C(e_\alpha(S_T)) = (1-\alpha)P(e_\alpha(S_T)),$$

where $e_\alpha(S_T)$ denotes the α-expectile of the risk neutral distribution, that we call simply implicit expectile. In other words, it is the value of the strike price $\bar{K}$ such that

$$\frac{C(\bar{K})}{P(\bar{K})} = \frac{1-\alpha}{\alpha}.$$

Implicit expectiles

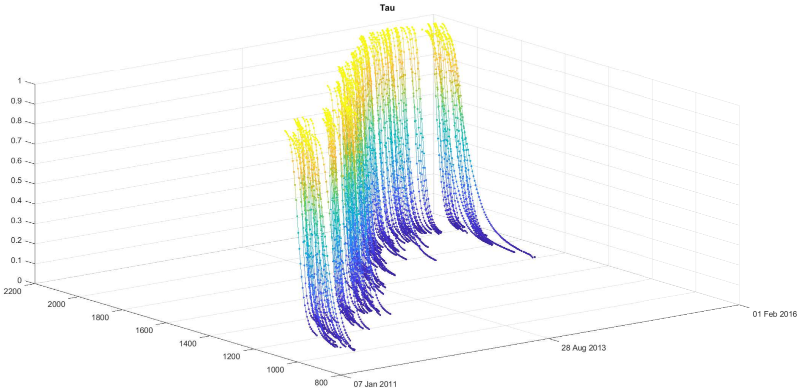
Since the quoted strikes are discrete, two approaches are possible:

- to interpolate option prices or Black-Scholes implied volatilities in order to derive prices for a continuum of strikes, and then solve numerically equation (D2) for each $\alpha \in (0, 1)$
- to compute the inverse expectile curve $\alpha(K)$ for quoted strikes

$$\alpha(K_i) := \frac{P(K_i)}{P(K_i) + C(K_i)}, \quad i = 1, \ldots, n.$$

The first approach may be necessary in relatively illiquid markets. The second approach does not require any model assumption or interpolation method; the values $\alpha(K_i)$ are exact and not subject to discretization or truncation errors.

Implicit expectiles



Three dimensional visualization of inverse expectile curves. For each day and each strike, the corresponding α is represented on the z-axis.

Interexpectile difference

In Bellini et al. (2018) we considered interexpectile differences, defined as

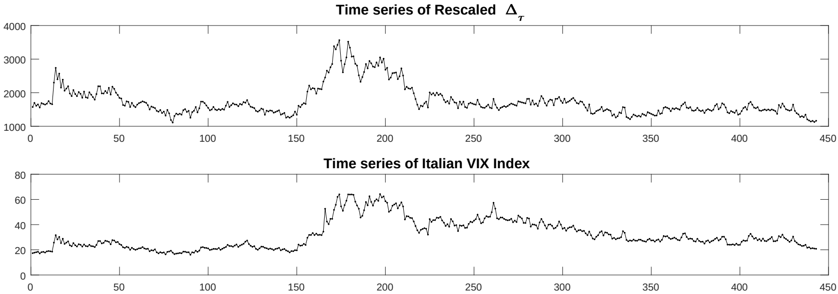
$$\Delta_\alpha(X) = e_\alpha(X) - e_{1-\alpha}(X), \quad \alpha \in (1/2, 1).$$

Interexpectile differences are natural variability measures similar to interquantile differences, with the theoretical advantage of being consistent with the convex order, in the sense that

$$X \leq_{cx} Y \Rightarrow \Delta_\alpha(X) \leq \Delta_\alpha(Y).$$

Our empirical analysis showed that interexpectile differences closely track the VIX Index, and on our dataset they seem also to have a significant forecasting power on future logreturns on various time horizons.

Empirical results



Graphical comparison between $\Delta_{0.75}$ and the Italian VIX index.

Forecasting power on future logreturns

	$\Delta_{0.75}$	Rescaled $\Delta_{0.75}$	Italian VIX	$e_{0.25}$	$e_{0.75}$
$R_{t,t+1}$	-6.833e-06**	-7.713e-06**	-1.727e-04	2.147e-09	-3.824e-08
	(-2.974)	(-3.137)	(9.483e-05)	(0.012)	(-0.214)
$R_{t,t+7}$	-4.031e-05***	-3.683e-05***	-0.0009704*	4.128e-08	-2.601e-07
	(-4.223)	(-3.540)	(-2.141)	(0.046)	(-0.286)
$R_{t,t+30}$	-1.203e-04*	-1.401e-04*	-0.003711	1.204e-06	3.205e-07
	(-2.288)	(-2.476)	(-1.479)	(0.234)	(0.062)
$R_{t,t+60}$	-2.773e-04*	-0.0002307	-0.009253	7.595e-06	4.651e-06
	(-3.169)	(-2.099)	(-1.826)	(0.539)	(0.321)
$R_{t,t+90}$	-0.0005915*	-0.0004034	-0.009919	3.005e-06	1.314e-06
	(-3.545)	(-1.476)	(-1.153)	(0.166)	(0.072)

Comparison of forecasting power of different volatility indexes by means of the significativity of a linear regression as in Elyasiani et al. (2016).

Risk contributions

The third and last financial application (Bellini et al., 2019) is the computation of Risk Parity portfolios using the expectiles as risk measures. For a fixed long only portfolio $x \in \mathbb{R}^n_{++}$, the total expectile is denoted by

$$\zeta_\alpha(x) := e_\alpha\left(\sum_{k=1}^{n} x_k L_k\right).$$

Expectile Risk Parity portfolios are defined by the requirement that the position in each asset equally contributes to the expectile of the portfolio. The standard approach for decomposing a positively homogeneous risk measure is Euler allocation (see e.g. Denault, 2001, Kalkbrener, 2005). Indeed if $\zeta_\alpha(x)$ is differentiable, then by Euler's theorem

$$\zeta_\alpha(x) = \sum_{k=1}^{n} x_k \frac{\partial \zeta_\alpha(x)}{\partial x_k}.$$

Risk contributions

It follows that the quantity $x_k \frac{\partial \zeta_\alpha(x)}{\partial x_k}$ can be interpreted as the total risk contribution of the position in asset k. In Emmer et al. (2015) it was proved that if the partial derivative of $\zeta_\alpha(x)$ with respect to x_k exists, then

$$\frac{\partial \zeta_\alpha(x)}{\partial x_k} = \frac{\alpha \mathbb{E}[L_k \mathbb{1}_{\{L > e_\alpha(L)\}}] + (1-\alpha)\mathbb{E}[L_k \mathbb{1}_{\{L \leq e_\alpha(L)\}}]}{\alpha \mathbb{P}(L > e_\alpha(L)) + (1-\alpha)\mathbb{P}(L \leq e_\alpha(L))},$$

where

$$L(x) = \sum_{k=1}^{n} x_k L_k.$$

For discrete distributions such those originating from historical scenarios the differentiability of $\zeta_\alpha(x)$ cannot be always guaranteed. However, Euler's theorem holds also for subdifferentiable functions under more general conditions (see e.g. Yang and Wei, 2008).

Risk contributions

Indeed, if we define

$$TRC_k^{e_\alpha}(x) := x_k \frac{\alpha \mathbb{E}[L_k \mathbb{1}_{\{L > e_\alpha(L)\}}] + (1-\alpha)\mathbb{E}[L_k \mathbb{1}_{\{L \leq e_\alpha(L)\}}]}{\alpha \mathbb{P}(L > e_\alpha(L)) + (1-\alpha)\mathbb{P}(L \leq e_\alpha(L))},$$

it always holds that

$$\sum_{k=1}^{n} TRC_k^{e_\alpha}(x) = e_\alpha(L).$$

It is possible to prove that nondifferentiability arises if and only if

$$\mathbb{P}(L = e_\alpha(L)) > 0,$$

i.e. iff there is a positive mass in the expectile. The impact of the lack of differentiability is that Euler allocations are no more uniquely defined; there is a subjective choice in the very definition of risk contributions, and as a result a Risk Parity portfolio for the chosen definition may not exist.

The additive case

Risk contributions do not admit in general an explicit espression.
A special case arises if the random vector $(L_1, \ldots, L_n)$ is such that the risk measure ρ is additive, in the sense that

$$\rho(L) = \sum_{k=1}^{n} x_k \rho(L_k),$$

since here clearly

$$TRC_k^{\rho}(x) = x_k \rho(L_k),$$

so the total risk contribution of the position in each asset coincides with its own risk measure. When the risk measure ρ is *VaR* or *CVaR*, it is well known that additivity holds if the random vector $(L_1, \ldots, L_n)$ is comonotonic. For expectiles, it is possible to prove that additivity holds if and only if

$$\mathbb{P}\left((X - e_\alpha(X))\,(Y - e_\alpha(Y)) \geq 0\right) = 1.$$

Risk Parity portfolios

Expectile Risk Parity portfolios are thus defined by

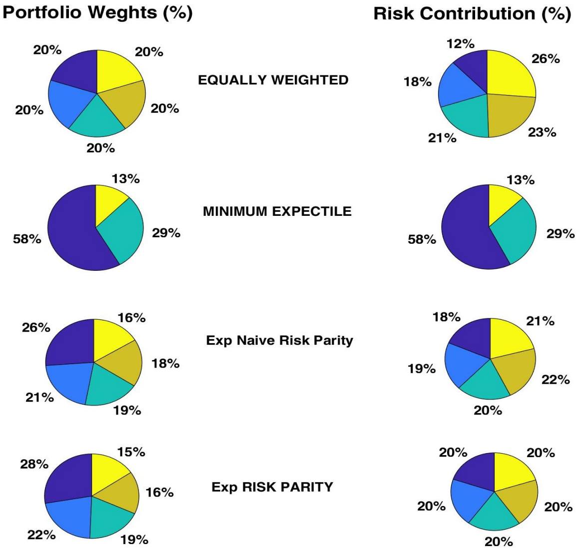
$$TRC_i^{e_\alpha}(x) = TRC_k^{e_\alpha}(x), \text{ for } i, k = 1, \ldots, n, \; i \neq k.$$

In order to find Risk Parity portfolios, we considered three approaches:

- "system-of-equations" formulations
- "log" formulations
- "least squares" formulations.

Before entering a detailed numerical comparison, we provide first a simple illustration based on the daily returns of 5 stocks taken from DJIA, from 16/02/1990 to 07/04/2016.

Expectile Risk Parity portfolios



Risk Parity portfolios: system-of-equations formulations

The most direct approach is to try to solve the nonlinear system

$$\begin{cases} \alpha \sum_{t=1}^{T} \left(\sum_{i=1}^{n} x_i L_{i,t} - \zeta_\alpha \right)_+ = (1-\alpha) \sum_{t=1}^{T} \left(\zeta_\alpha - \sum_{i=1}^{n} x_i L_{i,t} \right)_- \\ TRC_i^{Exp}(y) = \delta, \quad i = 1, \dots, n \end{cases} \qquad \textbf{(fsolve-many)}$$

or equivalently

$$\begin{cases} \alpha \sum_{t=1}^{T} \left(\sum_{i=1}^{n} x_i L_{i,t} - \zeta_\alpha \right)_+ = (1-\alpha) \sum_{t=1}^{T} \left(\zeta_\alpha - \sum_{i=1}^{n} x_i L_{i,t} \right)_- \\ TRC_i^{Exp}(x) = \frac{\zeta_\alpha}{n}, \quad i = 1, \dots, n \end{cases} \qquad \textbf{(fsolve-few)}$$

respectively in $n+2$ and $n+1$ variables.

Risk Parity portfolios: log formulations

A second approach (see e.g. Maillard, 2010, Spinu, 2013, Mausser and Romanko, 2018) is to solve the problem

$$\begin{cases} \min_{x \in \mathbb{R}^n_{++}} & \zeta_\alpha(x) \\ s.t. & \\ & \sum_{k=1}^n \ln x_k \geq c \end{cases}, \qquad \textbf{(LogConstr)}$$

where $c \in \mathbb{R}$ is an arbitrary constant. A variant is to put the log in the objective function (see Bai et al., 2016, Cesarone et al., 2018) and solve

$$\begin{cases} \min_{x \in \mathbb{R}^n_{++}} & \zeta_\alpha(x) - \lambda \sum_{k=1}^n \ln x_k \end{cases}, \qquad \textbf{(LogObjFun)}$$

for some $\lambda > 0$. Both problems admit a unique solution if the expectile of the global minimum expectile portfolio is strictly positive.

Risk Parity portfolios: least squares formulations

Finally, a third approach (following e.g. Maillard, 2010) is to consider directly the problem

$$\left\{ \min_{x \in \mathbb{R}^n_{++}, \sum x_k = 1} \sum_{i,k=1}^{n} \left(TRC_i^{e_\alpha}(x) - TRC_k^{e_\alpha}(x) \right)^2 \right. \qquad \textbf{(LS)}$$

An alternative least squares formulation (following e.g. Cesarone and Colucci, 2018) is to minimize the sum of squared deviations of relative risk contributions from $1/n$:

$$\left\{ \min_{x \in \mathbb{R}^n_{++}, \sum x_k = 1} \sum_{k=1}^{n} \left(\frac{TRC_k^{e_\alpha}(x)}{\zeta_\alpha(x)} - \frac{1}{n} \right)^2 \right. \qquad \textbf{(LSRel)}$$

Both problems are nonconvex and require the hypothesis that the expectile of the global minimum expectile portfolio is strictly positive.

Empirical analysis

In a numerical experiment on portfolios from the NASDAQ 100, we considered several values of α $(= 0.52, 0.6, 0.7, 0.8, 0.9, 0.95)$, n (=5, 10, 20, 82), and T $(= 50, 100, 200, 596)$. In the next slide we report

$$\mathbf{F}(\mathbf{x}) = \sum_{i,k=1}^{n} \left(TRC_i^{Exp}(x) - TRC_k^{Exp}(x) \right)$$

$$\mathbf{MeanAbsDev} = \frac{1}{n} \sum_{k=1}^{n} \left| \frac{TRC_k^{Exp}(x)}{\zeta_\alpha(x)} - \frac{1}{n} \right|$$

$$\mathbf{MaxAbsDev} = \max_{1 \le k \le n} \left| \frac{TRC_k^{Exp}(x)}{\zeta_\alpha(x)} - \frac{1}{n} \right|$$

and the running times on a workstation with Intel(R) Core(TM) i7-3520M CPU (2.9 GHz, 8 GB RAM) under Windows 10 Pro, using Matlab R2016b.

Empirical analysis

$\alpha = \mathbf{0.7}$; $n = 5$; $T = 100$; MinExp$= 0.45 \cdot 10^{-2}$					$\alpha = \mathbf{0.9}$; $n = 5$; $T = 100$; MinExp$= 2.00 \cdot 10^{-2}$			
	F(x)	**MeanAbsDev**	**MaxAbsDev**	time (secs.)	**F(x)**	**MeanAbsDev**	**MaxAbsDev**	time (secs.)
fsolve_few	$3.06\ 10^{-37}$	$2.22\ 10^{-17}$	$2.78\ 10^{-17}$	0.0	$1.90\ 10^{-6}$	$1.00\ 10^{-2}$	$2.50\ 10^{-2}$	0.0
fsolve_many	$3.06\ 10^{-37}$	$2.22\ 10^{-17}$	$2.78\ 10^{-17}$	0.0	$1.90\ 10^{-6}$	$1.00\ 10^{-2}$	$2.50\ 10^{-2}$	0.0
LogConstr	$4.32\ 10^{-14}$	$6.23\ 10^{-6}$	$1.46\ 10^{-5}$	64.6	$3.42\ 10^{-7}$	$4.66\ 10^{-3}$	$8.93\ 10^{-3}$	1.9
LogObjFun	$2.57\ 10^{-20}$	$5.34\ 10^{-9}$	$1.29\ 10^{-8}$	2.3	$1.90\ 10^{-6}$	$1.00\ 10^{-2}$	$2.50\ 10^{-2}$	1.4
LS	$1.11\ 10^{-19}$	$1.14\ 10^{-8}$	$2.45\ 10^{-8}$	1.6	$1.90\ 10^{-6}$	$1.00\ 10^{-2}$	$2.50\ 10^{-2}$	1.3
LSRel	$5.82\ 10^{-18}$	$8.48\ 10^{-8}$	$1.56\ 10^{-7}$	2.0	$3.34\ 10^{-7}$	$4.73\ 10^{-3}$	$8.38\ 10^{-3}$	1.8
$\alpha = \mathbf{0.7}$; $n = 10$; $T = 100$; MinExp$= 0.26 \cdot 10^{-2}$					$\alpha = \mathbf{0.9}$; $n = 10$; $T = 100$; MinExp$= 1.67 \cdot 10^{-2}$			
fsolve_few	$2.19\ 10^{-7}$	$7.23\ 10^{-3}$	$3.61\ 10^{-2}$	6.2	$1.20\ 10^{-5}$	$1.41\ 10^{-2}$	$3.96\ 10^{-2}$	0.0
fsolve_many	$2.72\ 10^{-8}$	$3.56\ 10^{-3}$	$9.17\ 10^{-3}$	0.1	$1.16\ 10^{-5}$	$1.39\ 10^{-2}$	$3.87\ 10^{-2}$	0.1
LogConstr	$9.51\ 10^{-8}$	$5.96\ 10^{-3}$	$1.62\ 10^{-2}$	6.1	$7.48\ 10^{-7}$	$4.02\ 10^{-3}$	$8.33\ 10^{-3}$	3.6
LogObjFun	$2.08\ 10^{-8}$	$2.40\ 10^{-3}$	$1.09\ 10^{-2}$	3.8	$3.79\ 10^{-7}$	$2.69\ 10^{-3}$	$7.30\ 10^{-3}$	2.1
LS	$2.08\ 10^{-8}$	$2.40\ 10^{-3}$	$1.09\ 10^{-2}$	2.4	$3.79\ 10^{-7}$	$2.69\ 10^{-3}$	$7.30\ 10^{-3}$	1.7
LSRel	$2.13\ 10^{-6}$	$2.02\ 10^{-2}$	$1.01\ 10^{-1}$	15.0	$4.57\ 10^{-7}$	$2.82\ 10^{-3}$	$7.31\ 10^{-3}$	2.3

Experimental results for solving the ExpRP models for $\alpha = 0.7, 0.9$, $n = 5, 10$, and $T = 100$.

Empirical analysis

$\alpha = \mathbf{0.7}$; $n = 20$; $T = 596$; MinExp$= 0.33 \cdot 10^{-2}$					$\alpha = \mathbf{0.9}$; $n = 20$; $T = 596$; MinExp$= 1.57 \cdot 10^{-2}$			
	F(x)	**MeanAbsDev**	**MaxAbsDev**	time (secs.)	**F(x)**	**MeanAbsDev**	**MaxAbsDev**	time (secs.)
fsolve_few	$1.08\ 10^{-7}$	$2.77\ 10^{-3}$	$1.41\ 10^{-2}$	0.2	$2.00\ 10^{-3}$	$2.93\ 10^{-3}$	$9.29\ 10^{-3}$	0.1
fsolve_many	$1.08\ 10^{-7}$	$2.77\ 10^{-3}$	$1.41\ 10^{-2}$	0.2	$2.01\ 10^{-3}$	$2.93\ 10^{-3}$	$9.34\ 10^{-3}$	0.1
LogConstr	$2.75\ 10^{-7}$	$4.06\ 10^{-3}$	$2.30\ 10^{-2}$	13.8	$5.68\ 10^{-8}$	$4.71\ 10^{-4}$	$1.54\ 10^{-3}$	1644.5
LogObjFun	$1.06\ 10^{-5}$	$2.55\ 10^{-2}$	$9.94\ 10^{-2}$	2611.1	$7.38\ 10^{-6}$	$5.86\ 10^{-3}$	$1.57\ 10^{-2}$	2470.4
LS	$1.88\ 10^{-8}$	$7.83\ 10^{-4}$	$6.75\ 10^{-3}$	581.4	$3.74\ 10^{-7}$	$9.91\ 10^{-4}$	$6.22\ 10^{-3}$	308.4
LSRel	$1.08\ 10^{-8}$	$1.01\ 10^{-3}$	$3.15\ 10^{-3}$	22.4	$1.88\ 10^{-8}$	$2.72\ 10^{-4}$	$1.11\ 10^{-3}$	52.7
$\alpha = \mathbf{0.7}$; $n = 82$; $T = 596$; MinExp$= 0.20 \cdot 10^{-2}$					$\alpha = \mathbf{0.9}$; $n = 82$; $T = 596$; MinExp$= 1.38 \cdot 10^{-2}$			
fsolve_few	$1.71\ 10^{-6}$	$2.54\ 10^{-3}$	$1.29\ 10^{-2}$	20.0	$3.58\ 10^{-8}$	$6.84\ 10^{-5}$	$4.85\ 10^{-4}$	21.1
fsolve_many	$1.71\ 10^{-6}$	$2.54\ 10^{-3}$	$1.29\ 10^{-2}$	2.6	$3.99\ 10^{-8}$	$7.17\ 10^{-5}$	$5.32\ 10^{-4}$	1.4
LogConstr	$4.03\ 10^{-7}$	$1.01\ 10^{-3}$	$1.32\ 10^{-2}$	124.5	$1.12\ 10^{-8}$	$4.26\ 10^{-5}$	$3.50\ 10^{-4}$	98.9
LogObjFun	$1.34\ 10^{-4}$	$1.07\ 10^{-2}$	$1.62\ 10^{-1}$	2724.2	$2.55\ 10^{-3}$	$1.06\ 10^{-2}$	$1.86\ 10^{-1}$	3264.3
LS	$2.25\ 10^{-8}$	$2.09\ 10^{-4}$	$3.69\ 10^{-3}$	999.1	$1.17\ 10^{-8}$	$4.30\ 10^{-5}$	$3.45\ 10^{-4}$	656.7
LSRel	$1.68\ 10^{-6}$	$2.53\ 10^{-3}$	$1.25\ 10^{-2}$	374.8	$1.08\ 10^{-8}$	$4.30\ 10^{-5}$	$3.21\ 10^{-4}$	134.1

Experimental results for solving the ExpRP models for $\alpha = 0.7, 0.9$, $n = 20, 82$, and $T = 596$.

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